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Normal cone approximation and offset shape isotopy

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ABSTRACT

This work addresses the problem of the approximation of the normals of the offsets of general compact sets in Euclidean spaces. It is proven that for general sampling conditions, it is possible to approximate the gradient vector field of the distance to general compact sets. These conditions involve the μ-reach of the compact set, a recently introduced notion of feature size. As a consequence, we provide a sampling condition that is sufficient to ensure the correctness up to isotopy of a reconstruction given by an offset of the sampling. We also provide a notion of normal cone to general compact sets that is stable under perturbation.

1. Introduction

1.1. Motivation

Let K' be a finite set of points measured, possibly with some noise, on a physical object K. Given K' as *input*, is it possible to infer some reliable information on first order properties such as tangent planes or sharp edges, of the boundary of K? We consider here the case when the approximation K' of K has an error bounded under the Hausdorff distance. In other words we only assume that $d_H(K, K') < \varepsilon$ which means that any point of K' lies within a distance ε of some point of K and symmetrically, any point of K lies within a distance ε of a point of K'. The question is of primary interest in surface reconstruction applications. More generally, in the context of geometric processing, we would like to be able to extrapolate to a large class of non-smooth compact sets, including finite points samples and meshes, the usual notions of tangent plane or normal cones. Our goal here is to explore the notion of tangent plane first through the generalized gradient of the distance function (Section 3) and second through the Clarke Gradient of the distance function (Section 5), which brings also informations on concaves sharp edges.

1.2. Previous work on smooth manifolds

When K' is sampled exactly: $K' \subset K$, on a smooth boundary, it has been proved [2,3], that the normals to K can be estimated from the *poles*: for each point sample $q \in K'$, its pole is the Voronoi vertex farthest from q on the boundary of the Voronoi cell of q. In [12] this Voronoi based approach has been extended to the approximation of normals and feature lines from *noisy* sampling of a smooth manifold by considering only the poles corresponding to sufficiently large Delaunay balls.

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1.3. Reconstruction of "sufficiently regular" non-smooth objects from samples

In [5,7], the authors have considered the problem of recovering the topology of a compact set K given a sampling K' without any smoothness assumption on K.

In the same manner as the resolution of a microscope constrains the minimal size of observable details, any topological feature (such as a connected component or a tunnel for example) of a compact set K which would be small with respect to ε can certainly not be "reliably detected" from the knowledge of a sample K' with Hausdorff distance bounded by ε. A realistic measure of the topology should consider only the "topological information observable at the scale ε": in the context of [5,7], this has lead to consider topological features which are stable under sufficiently large offsets. Note that *topological persistence* [8] is an algebraic counterpart of this notion of stable topology.

The problem of the reconstruction, from a set of measured points, of a geometric numerical model carrying the same topology as the sampled object has been addressed previously for smooth manifolds [1,19], for which the sampling condition is related to the distance to the medial axis of K. The main contribution of [5] is to give a sampling condition for non-smooth objects, through the notion of *critical function* which encodes the regularity of the compact set boundary at different "scales".

When it is reasonable to assume some regularity conditions on the object's boundary, which can be formally expressed through lower bounds on the critical function, it is possible to recover the object's topology from a sufficiently dense and accurate sampling. By contrast, if we make no assumption about the regularity of the measured object K, it is still possible to decide some guaranteed topological information, not about the object K itself of course, but about offsets of K.

1.4. Contributions

The aim of the present work is to apply the previous approach, which has been successful for the retrieval of topological information, to the determination, beyond the topology, of first order information, which include tangents planes or the detection of sharp edges. Note that classical "exact" definitions of first order geometric informations such as tangent planes on surfaces, are not preserved in general by Hausdorff approximations. In other words, they are in general destroyed by arbitrary small perturbations (small for Hausdorff distance) of the object boundary. For example, a finite set sampled "near" the boundary of a smooth shape "contains" some information about the shape boundary tangent plane, but has no tangent plane in the usual sense. Still if one consider a d-offset of the point sample, that is a union of spheres of radius d centered on the points, the tangent planes to the offset boundary may bring some meaningful tangency informations about the initial shape. Following this simple idea and using properties of the distance function to compact sets developed in [5] we propose to introduce "stable" quantities that extend usual exact first order differential quantities. These quantities are preserved by small Hausdorff distance perturbation of the object: from this perspective, they can be "really observed" and carry more reality than their classical "exact and ideal" counterpart. These stable quantities are generalization of first order differential properties of surfaces. They apply to a large class of compact sets, which suggest applications for meshes and point clouds modeling. For smooth manifolds, our quantities coincide, in the limit, with usual definitions of first order tangent affine manifold.

The paper follows two complementary approaches. Both are necessary because they bring two distinct stability results, none being a corollary of the other. The first stability result given in Section 3 applies to the generalized gradient of the distance function and is necessary to get the isotopy result of Section 4. On the other hand, the second stability result (Section 5) applies to the Clarke gradient of the distance function which brings more information than the generalized gradient, in particular it may allow to recognize concave edges. In some sense, the first stability result uses weaker conditions (this is why the second stability result can not be used in the proof of isotopy in Section 4). On another hand the second stability result ensures a better convergence of the estimation of the gradient direction with respect to the Hausdorff distance (error bounded by $O(\epsilon^{\frac{1}{2}})$ instead of $O(\epsilon^{\frac{1}{4}})$) for the gradient estimation near a smooth surfaces).

1.5. Outline

Section 2 gives the necessary background notions on the distance function and its generalized gradient. Section 3 and in particular Corollary 3.2 gives a first stability property of the generalized gradient with respect to perturbations of the compact sets bounded in Hausdorff distance. This property bounds the maximal angular deviation between the gradient of the distance functions to two compact sets K and K'. An important consequence of this theorem is Theorem 4.2 which asserts the isotopy between the offsets of the compact set and its sampling with almost the same sampling conditions as in the main theorem in [5]. Section 5 introduces a stability theorem on the Clarke Gradient of the distance function. The stable quantity is a kind of "interval Clarke Gradient": to be more precise, it is the convex hull of the union of the values taken by the Clarke gradient in a ball. From this stability theorem, one introduces (Section 6), a *normal cone at a given scale*, which is a stable generalization of first order differential properties, defined at any point on or nearby a compact set.