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Plane geodesic spanning trees, Hamiltonian cycles, and perfect matchings in a simple polygon ☆

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ABSTRACT

Let S be a finite set of points in the interior of a simple polygon P. A *geodesic graph*, $G_P(S, E)$, is a graph with vertex set S and edge set E such that each edge $(a, b) \in E$ is the shortest geodesic path between a and b inside P. G_P is said to be *plane* if the edges in E do not cross. If the points in S are colored, then G_P is said to be *properly colored* provided that, for each edge $(a, b) \in E$, a and b have different colors. In this paper we consider the problem of computing (properly colored) plane geodesic perfect matchings, Hamiltonian cycles, and spanning trees of maximum degree three.

1. Introduction

Let S be a set of n points in the interior of a simple polygon P with m vertices. For two points a and b in the interior of P, the *geodesic*, $\pi(a, b)$, is defined to be the shortest path between a and b in the interior of P. A *geodesic graph*, $G_P(S, E)$, is a topological graph with vertex set S and edge set E such that each edge $(a, b) \in E$ is the geodesic $\pi(a, b)$ in P. If P is a convex polygon, then G_P is a straight-line geometric graph.

Problems related to geodesic graphs have been of interest in recent years. Many problems and structures related to the Euclidean plane have been generalized to the geodesic setting, e.g., convex hull [10,19], furthest-point Voronoi diagram [5,6,16], ham-sandwich cut [8], center of a point set [2,15,17]. In this paper we study Hamiltonian cycle, perfect matchings, and spanning trees in geodesic graphs.

Let π_1 and π_2 be two, possibly self-intersecting, curves. We say that π_1 and π_2 *cross* if by traversing π_1 from one of its endpoints to the other endpoint, it intersects π_2 and switches from one side of π_2 to the other side [19].[1] We say that π_1 and π_2 are *non-crossing* if they do not cross. Two non-crossing curves can share an endpoint or can "touch" each other. If π_1 and π_2 are geodesics in a simple polygon, then they can intersect only once. They may have common line segments, but once they break apart, they do not meet again. See Fig. 1. A geodesic graph is said to be *plane* if the edges in E are pairwise non-crossing.

If the points in S are colored, then a geodesic graph G_P is said to be *properly colored* provided that, for each edge $(a, b) \in E$, a and b have different colors. For simplicity, in this paper we refer to a properly colored graph as a "colored graph". Let $\{S_1, \dots, S_k\}$, where $k \geq 2$, be a partition of S. Let $K_P(S_1, \dots, S_k)$ be the complete multipartite geodesic graph

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1 Tousssaint [19] refers to this configuration as a "proper crossing".

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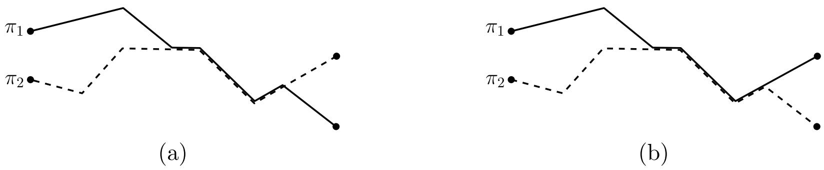

Fig. 1. (a) Two crossing geodesics, and (b) two non-crossing geodesics.

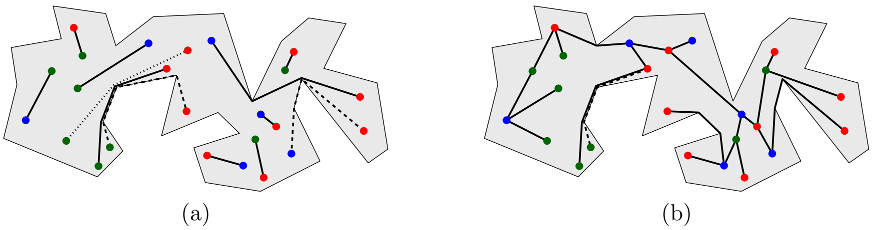

Fig. 2. (a) A plane colored geodesic matching, and (b) a plane colored geodesic 3-tree. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

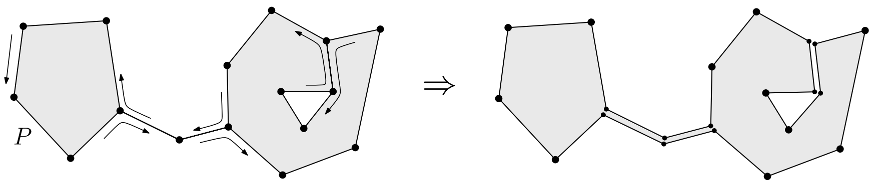

Fig. 3. A weakly simple polygon P whose interior is shaded, together with the corresponding simple polygon after perturbation.

on S that has an edge between every point in S_i and every point in S_j, for all $1 \le i < j \le k$. Imagine the points in S to be colored, such that all the points in S_i have the same color, and for $i \ne j$, the points in S_i have a different color from the points in S_j. We say that S is a *k-colored* point set. Any colored geodesic graph, $G_P(S, E)$, is a subgraph of $K_P(S_1, \ldots, S_k)$.

If G_P is a perfect matching, a spanning tree, or a Hamiltonian cycle, we call it a *geodesic matching*, a *geodesic tree*, or a *geodesic Hamiltonian cycle*, respectively. A *colored matching* is a geodesic matching in $K_P(S_1, \ldots, S_k)$. Similarly, a *colored tree* (resp. a *colored Hamiltonian cycle*) is a geodesic tree (resp. geodesic Hamiltonian cycle) in $K_P(S_1, \ldots, S_k)$. A *plane colored matching* is a colored matching in $K_P(S_1, \ldots, S_k)$ that is non-crossing. Similarly, a *plane colored tree* (resp. a *plane colored Hamiltonian cycle*) is a colored tree (resp. colored Hamiltonian cycle) that is non-crossing. Given a (colored) point set S in the interior of a simple polygon P, we consider the problem of computing a plane colored geodesic matching, a plane colored geodesic 3-tree, and a plane geodesic Hamiltonian cycle in $K_P(S_1, \ldots, S_k)$. A *t-tree* is a tree of maximum degree t. See Fig. 2.

1.1. Preliminaries

We say that a set S of points in the pale is in *general position* if no three points of S are collinear. Moreover, we say that a set S of points in a simple polygon is *geodesically in general position* provided that, for any two points a and b in S, $\pi(a, b)$ does not contain any point of $S \setminus \{a, b\}$.

Toussaint [19] defined weakly-simple polygons—as a generalization of simple polygons—because in many situations concerned with geodesic paths the regions of interest are not simple but weakly-simple. A *weakly simple polygon* is defined as a closed polygonal chain $P = (p_1, \ldots, p_m)$, possibly with repeated vertices, such that every pair of distinct vertices of P partitions P into two non-crossing polygonal chains [19]. Alternatively, a closed polygonal chain P is weakly simple if its vertices can be perturbed by an arbitrarily small amount such that the resulting polygon is simple. See Fig. 3. From the computational complexity point of view, almost all data structures and algorithms developed for simple polygons work for weakly simple polygons with only minor modifications that do not affect the time or space complexity bounds. Hereafter, we consider a weakly simple polygon to be a simple polygon.

For two points a and b in the interior of a simple polygon P, $\pi(a, b)$ consists of a sequence of straight-line segments. We refer to a and b as the *external vertices* of $\pi(a, b)$, and refer to the other vertices of $\pi(a, b)$ as *internal vertices*. Moreover, we refer to the line segment(s) of $\pi(a, b)$ that are incident on a or b as the *external segments* and the other segments as *internal segments*. In the special case where $\pi(a, b)$ is a straight-line segment, $\pi(a, b)$ does not have any internal vertex nor any internal segment.

Observation 1. *The set of internal vertices of any geodesic in a simple polygon P is a subset of the reflex vertices of P.*