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# Discrete Voronoi games and $\epsilon$-nets, in two and three dimensions

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A B S T R A C T

The one-round discrete Voronoi game, with respect to an $n$-point user set $U$, consists of two players Player 1 ($\mathcal{P}_1$) and Player 2 ($\mathcal{P}_2$). At first, $\mathcal{P}_1$ chooses a set of facilities $F_1$ following which $\mathcal{P}_2$ chooses another set of facilities $F_2$, disjoint from $F_1$. The payoff of $\mathcal{P}_2$ is defined as the cardinality of the set of points in $U$ which are closer to a facility in $F_2$ than to every facility in $F_1$, and the payoff of $\mathcal{P}_1$ is the difference between the number of users in $U$ and the payoff of $\mathcal{P}_2$. The objective of both the players in the game is to maximize their respective payoffs. In this paper we study the one-round discrete Voronoi game where $\mathcal{P}_1$ places $k$ facilities and $\mathcal{P}_2$ places one facility. We denote this game as $\mathrm{VG}(k,1)$. Although the optimal solution of this game can be found in polynomial time, the polynomial has a very high degree. In this paper, we focus on achieving approximate solutions to $\mathrm{VG}(k,1)$ with significantly better running times. We provide a constant-factor approximate solution to the optimal strategy of $\mathcal{P}_1$ in $\mathrm{VG}(k,1)$ by establishing a connection between $\mathrm{VG}(k,1)$ and weak $\epsilon$-nets. To the best of our knowledge, this is the first time that Voronoi games are studied from the point of view of $\epsilon$-nets.



## 1. Introduction

Facility location is a sub-field of operations research and computational geometry that focuses on the optimal placement of facilities, subject to a set of constraints. One of the most well-known facility location problems is the computation of the minimum enclosing disk. Given a set $U$ of $n$ points in the plane, compute the smallest circle that encloses $U$. There is a $O(n)$ time algorithm by Megiddo [24] to solve this problem as well as a $O(n)$ expected time algorithm by Welzl [28]. For an extensive discussion on geometric variants of the facility location problem, refer to Fekete et al. [16].

*Competitive* facility location (or *competitive spatial modelling*) is concerned with the strategic placement of facilities by competing market players, subject to a set of constraints. In this setting, each facility has its *service zone*, consisting of the set of users it serves. Different metrics can be used to determine the users served by a given facility. In general, the service zone of a player does not have to be connected. In the continuous setting, the objective of each player is to place a set of facilities in order to maximize the total area of its service zone. As for in the discrete setting, the objective of each player is to place a set of facilities in order to maximize the total number of users present in its service zone. The study of competitive facility

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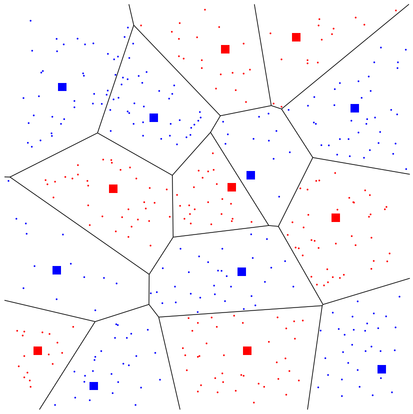

**Fig. 1.** A set $U$ of $n$ of users, denoted by small points, among two competing market players denoted by red and blue squares. When we subdivide the space according to the nearest-neighbor rule, we get the Voronoi diagram of the set of facilities. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

location started by the work of Hotelling [19] back in 1929. The discrete setting was introduced by Eaton and Lipsey [12]. Numerous variants of facility location problems have been studied (refer to [14,15,17] for comprehensive surveys).

*Voronoi games*, introduced by Ahn et al. [1] (for the one-dimensional case), and by Cheong et al. [10] (for two- and higher-dimensional cases), consist in the following competitive facility location problem. Two players alternately place one facility in a bounded region $\mathcal{B} \subset \mathbb{R}^d$, until each of them has placed a given number of points. Then we subdivide $\mathcal{B}$ according to the nearest-neighbor rule. The player whose facilities control the larger volume wins. In the discrete setting, introduced by Banik et al. [4,3], the players are given a set $U$ of $n$ users in $\mathbb{R}^d$ (refer to Fig. 1). Then, as in the continuous setting, the players alternately place one facility, until each of them has placed a given number of points. Then we subdivide the space according to the nearest-neighbor rule. The player whose service zone contains the largest number of users wins. To *solve* a Voronoi game corresponds to finding an optimal strategy for each player. In this paper, we establish a connection between Voronoi games and weak $\varepsilon$-nets, which leads to general bounds on the scores of the players for discrete Voronoi games.

### 1.1. Preliminaries and related work

In this section, we introduce the notation we use throughout the paper, we define the variants of Voronoi games we study and we explain how these variants compare to existing ones. Consider a set $U$ of $n$ users and let $F$ be a finite set of facilities. In this paper, we identify each facility with its location. A user $u \in U$ is said to be served by a facility $f \in F$ if $f$ is the facility that is closest to $u$. For every facility $f \in F$, we define the *service zone of* $f$, denoted $U(f, F)$, as the set of users from $U$ that are served by $f$. The *discrete Voronoi game* is a competitive facility location problem involving two players $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. The players $\mathcal{P}_1$ and $\mathcal{P}_2$ alternately place two disjoint sets of facilities $F_1$ and $F_2$, respectively. A user $u \in U$ is said to be served by $\mathcal{P}_1$ (respectively by $\mathcal{P}_2$) if $u$ belongs to the service zone of at least one facility placed by $\mathcal{P}_1$ (respectively by $\mathcal{P}_2$). In such a case, we say that $u$ is in the service zone of $\mathcal{P}_1$ (respectively of $\mathcal{P}_2$). The *payoff* $\mathcal{V}(F_1, F_2)$ *of* $\mathcal{P}_2$ (or the *value of the game*) is defined as the cardinality of the set of users from $U$ that belong to the service zone of $\mathcal{P}_2$. More formally,

$$\mathcal{V}(F_1, F_2) = \left| \bigcup_{f \in F_2} U(f, F_1 \cup F_2) \right|.$$

Similarly, the payoff of $\mathcal{P}_1$ is

$$|U| - \mathcal{V}(F_1, F_2) = \left| \bigcup_{f \in F_1} U(f, F_1 \cup F_2) \right|.$$

If $\mathcal{V}(F_1, F_2) > |U| - \mathcal{V}(F_1, F_2)$, we say that $\mathcal{P}_2$ wins. If $\mathcal{V}(F_1, F_2) < |U| - \mathcal{V}(F_1, F_2)$, we say that $\mathcal{P}_1$ wins. Otherwise, it is declared a tie. In the *one-round Voronoi Game*, $\mathcal{P}_1$ places all its facilities after which $\mathcal{P}_2$ places all its facilities. If $|F_1| = |F_2| = k$, the *k-round Voronoi Game* corresponds to a Voronoi game where the players alternately place their facilities one at a time (refer to [5]).

Let us define the *One-Round Discrete Voronoi Game*.

**Definition 1** *(One-Round Discrete Voronoi Game VG$(k, l)$).* Let $U$ be a set of $n$ users and $\mathcal{P}_1$ and $\mathcal{P}_2$ be two players. Initially, $\mathcal{P}_1$ chooses a set $F_1$ of $k$ locations in $\mathbb{R}^d$ for its facilities. Then $\mathcal{P}_2$ chooses a set $F_2$ of $l$ locations in $\mathbb{R}^d$ for its facilities, where $F_1 \cap F_2 = \emptyset$.