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On optimal disc covers and a new characterization of the Steiner center

Yael Yankelevsky *, Alfred M. Bruckstein

Department of Computer Science, Technion – Israel Institute of Technology, Haifa 32000, Israel

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ABSTRACT

Given N points in the plane $P_1, P_2, \dots, P_N$ and a location Ω, the union of discs with diameters $[\Omega P_i]$, $i = 1, 2, \dots, N$ covers the convex hull of the points. The location Ω_s minimizing the area covered by the union of discs, is shown to be the Steiner center of the convex hull of the points. Similar results for d-dimensional Euclidean space are conjectured.

1. Introduction

In this paper we discuss a sphere coverage problem and, in this context, we propose an optimal coverage criterion defining a center for a given set of points in space.

Suppose that a constellation of N points $\{P_1, P_2, \dots, P_N\}$ in $\mathbb{R}^d$ (the d-dimensional Euclidean space) is given. An arbitrary point $\Omega \in \mathbb{R}^d$ is selected and the spheres $S_{P_i}(\Omega)$, having $[\Omega P_i]$ as diameters, are defined. Hence the centers of $S_{P_i}(\Omega)$ are at $\frac{1}{2}(\Omega + P_i)$ and their radii are $\frac{1}{2}\|\Omega - P_i\|$, $i = 1, 2, \dots, N$.

Consider the union of these spheres $S_{P_i}(\Omega)$, their surface "anchored" at Ω. First we prove that the resulting d-dimensional shape always covers the convex hull $CH\{P_1, P_2, \dots, P_N\}$ of the given points, hence its volume exceeds the volume of this convex hull for all $\Omega \in \mathbb{R}^d$. This leads to the following natural question: what is the location Ω^* which minimizes the excess (or overflow) volume and hence the total volume of the shape, $\Sigma_{(\Omega)} = \bigcup_{i=1}^{N} S_{P_i}(\Omega)$?

Such a location, we claim, would be a natural candidate as a "center" for the constellation of points $\{P_1, P_2, \dots, P_N\}$.

The problem of determining the point that gives the tightest cover with spheres, minimizing the excess volume beyond the convex hull, is solved here for the planar case (i.e. $d = 2$). An illustration of this problem is presented in Fig. 1. The result is the following: the optimal location Ω^*, is the so called Steiner center of the convex hull of the given points $\{P_1, P_2, \dots, P_N\} \in \mathbb{R}^2$. The Steiner center is a weighted centroid of the vertices of a convex polygon, the weights being proportional to the exterior angles at the vertices (see Fig. 2). Hence, the Steiner center Ω_s of a convex polygon $[V_1 V_2 \dots V_k]$ is also characterized as the point that yields the tightest disc cover with discs having $[\Omega_s V_j]$ as diameters ($j = 1, 2, \dots k$).

For the d-dimensional case we conjecture that a similar result holds, however a proof is yet to be found. Some numerical simulations that were performed in 3D seem to confirm this conjecture.

* Corresponding author.
E-mail address: yaelyan@tx.technion.ac.il (Y. Yankelevsky).

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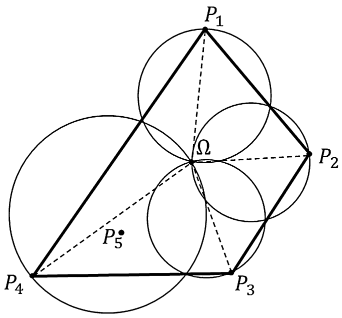

Fig. 1. Illustration of the problem in 2 dimensions: For the set of points $P_1, \ldots, P_5$, the (highlighted) convex hull polygon is defined by the vertices $P_1, \ldots, P_4$. These vertices define 4 discs anchored at the arbitrary point Ω.

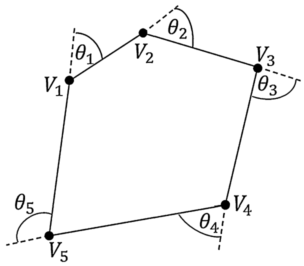

Fig. 2. External turn angles.

1.1. Centers for point constellations

Finding meaningful centers for a collection of data points is a fundamental geometric problem in various data analysis and operation research/facility location applications.

One of the interesting centers is the Steiner point (also known as the Steiner curvature centroid). The Steiner point of a convex polygon in $\mathbb{R}^2$, is defined as the weighted centroid (i.e. center of mass) of the system obtained by placing a mass equal to the magnitude of the exterior angle at each vertex [6]. The traditional characterization is therefore

$$\Omega_s = \arg\min_{\Omega} \sum_{i=1}^{k} \theta_i d^2(V_i, \Omega) \tag{1}$$

yielding explicitly

$$\Omega_s = \frac{1}{2\pi} \sum_{i=1}^{k} \theta_i V_i \tag{2}$$

where $d(V_i, \Omega)$ is the Euclidean distance from V_i to Ω and θ_i are the external turn angles at the vertices V_i of the convex polygon, that sum to 2π (see Fig. 2).

Another characterization of the Steiner center is by projections [4]. Let P_i^θ denote the projection of the point P_i on the unit vector $u_\theta = (\cos\theta, \sin\theta)$:

$$P_i^\theta = u_\theta < P_i, u_\theta > \tag{3}$$

then the Steiner center is defined as:

$$\Omega_s = \frac{1}{\pi} \int_0^\pi u_\theta \left(\min_i |P_i^\theta| + \max_i |P_i^\theta| \right) d\theta \tag{4}$$

Furthermore, the Steiner center Ω_s of a convex shape has some very interesting properties, the nicest one being its linearity with respect to Minkowski addition. Hence, if K_1 and K_2 are two convex sets in $\mathbb{R}^d$, we have that

$$\Omega_s(K_1 \oplus K_2) = \Omega_s(K_1) + \Omega_s(K_2) \tag{5}$$

where ⊕ stands for vector addition, i.e.