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An almost optimal algorithm for Voronoi diagrams of non-disjoint line segments ☆

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ABSTRACT

This paper presents an almost optimal algorithm that computes the Voronoi diagram of a set S of n line segments that may intersect or cross each other. If there are k intersections among the input segments in S, our algorithm takes $O(n\alpha(n)\log n + k)$ time, where $\alpha(\cdot)$ denotes the inverse of the Ackermann function. The best known running time prior to this work was $O((n+k)\log n)$. Since the lower bound of the problem is shown to be $\Omega(n\log n + k)$ in the worst case, our algorithm is worst-case optimal for $k = \Omega(n\alpha(n)\log n)$, and is only a factor of $\alpha(n)$ away from any optimal-time algorithm, which is still unknown. For the purpose, we also present an improved algorithm that computes the medial axis or the Voronoi diagram of a polygon with holes.

1. Introduction

There is no doubt that the Voronoi diagram is one of the most fundamental and the most well studied structure in computational geometry. Voronoi diagrams and their variations play an important role not only in computer science but also in many other fields in engineering and sciences, finding a lot of applications. For a comprehensive survey, we refer to Aurenhammer and Klein [2] or to a book by Okabe et al. [18].

In this paper, we are interested in the Voronoi diagram of line segments in the plane. As one of the most popular variants of the ordinary Voronoi diagram, the line segment Voronoi diagram has been extensively studied in the computational geometry community, finding lots of applications in computer graphics, pattern recognition, motion planning, shape representation, and NC machining [11,14,17]. Also, computing line segment Voronoi diagram is used as a frequent subroutine of algorithms for more complex structures [3,8]. For the set of line segments that are disjoint or may intersect only at their endpoints, a variety of optimal $O(n\log n)$-time algorithms that compute the diagram are known. For example, Kirkpatrick [14], Lee [17], and Yap [21] presented divide-and-conquer algorithms, Fortune [10] presented a plane sweep algorithm, and a pure abstract approach to Voronoi diagrams by Klein [15] is also applied to yield an optimal time algorithm [16].

However, only few researches consider line segments that may intersect or cross each other freely. Let S be a set of n arbitrary line segments in the plane and k be the number of intersecting pairs of the segments in S. Karavelas [12] presented an $O((n+k)\log^2 n)$ time algorithm that computes the Voronoi diagram of S in a robust way. In fact, one can easily achieve the time bound $O((n+k)\log n)$ for computing the Voronoi diagram of S as follows: first, specify all the intersection points among the segments in S and consider the set S' of sub-segments obtained by cutting the original segments in S at the

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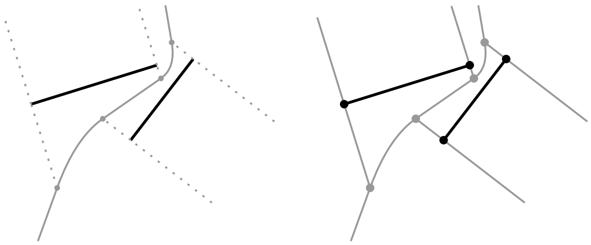

Fig. 1. (a) The Voronoi diagram of two line segments. (b) The Voronoi diagram of six sites: two open line segments and four of their endpoints. Sites are depicted as black segments or dots, and Voronoi edges and vertices are as gray segments and dots.

intersection points. Then, S' consists of at most $n+2k$ line segments that can intersect only at endpoints, so we can apply any of the above existing algorithms. On the other hand, it is not difficult to see that the lower bound of the problem of computing the Voronoi diagram of line segments is $\Omega(n\log n + k)$ in the worst case.

In this paper, we present an almost optimal algorithm that computes the Voronoi diagram of line segments. Our algorithm takes $O(n\alpha(n)\log n + k)$ time, where $\alpha(n)$ denotes the functional inverse of the Ackermann function. Since the lower bound is shown to be $\Omega(n\log n + k)$, our algorithm is only a factor of $\alpha(n)$ away from the optimal running time, and is optimal for large $k = \Omega(n\alpha(n)\log n)$. To our best knowledge, prior to our result, there was no known algorithm better than the simple $O((n+k)\log n)$-time algorithm.

In order to achieve our main result, we also consider an interesting special case where S forms a polygon. In this case, the Voronoi diagram of S is closely related to the *medial axis* of S [9,17]. When S forms a simple polygon, it is known that its Voronoi diagram and medial axis can be computed in linear time by Chin et al. [9]. In this work, we extend their result into more general form of polygons, namely, *weakly simple polygons* and *polygonal domains*. In particular, we devise an $O(m\log(m+t)+t)$-time algorithm that computes the Voronoi diagram or the medial axis of a given polygonal domain, where m denotes the total number of vertices of its holes and t denotes the number of vertices of its outer boundary. Note that our algorithm is strictly faster than any $O(n\log n)$ time algorithm when t is relatively larger than m. We exploit this algorithm for a polygonal domain as a subroutine of the $O(n\alpha(n)\log n + k)$-time algorithm that computes the Voronoi diagram of non-disjoint line segments.

The remaining of the paper is organized as follows: After introducing some preliminaries in Section 2, we present our algorithm that computes the Voronoi diagram of a polygonal domain in Section 3. Then, Section 4 is devoted to describe and analyze our algorithm that computes the Voronoi diagram of line segments.

2. Preliminaries

Throughout the paper, we use the following notations: For a subset $A \subset \mathbb{R}^2$, we denote by ∂A the boundary of A with the standard topology. For any two points $p \in \mathbb{R}^2$ and $q \in \mathbb{R}^2$, let $\overline{pq}$ denote the line segment between p and q.

2.1. Voronoi diagrams of line segments

Let S be a set of n line segments in the plane $\mathbb{R}^2$. The *Voronoi diagram* $\mathsf{VD}(S)$ of S is a subdivision of the plane $\mathbb{R}^2$ into *Voronoi regions* $R(s,S)$, defined to be

$$R(s,S) := \bigcap_{s' \in S\setminus\{s\}} \{x \in \mathbb{R}^2 \mid d(x,s) < d(x,s')\},$$

where $d(x,s)$ denotes the Euclidean distance from point x to segment s.

As done in the literature [1,21], we regard each segment $s \in S$ as three distinct *sites*: the two endpoints of s and the relative interior of s. We thus assume that the set S is implicitly the set of points and open line segments that form n closed line segments. See Fig. 1. Note that each Voronoi edge of $\mathsf{VD}(S)$ is then either a straight or parabolic segment. The Voronoi vertices of $\mathsf{VD}(S)$ are distinguished into two kinds: those of one kind are proper vertices which are equidistant points from three distinct sites, while those of the other kind are simply the intersection points of S at which at least three Voronoi edges meet. Fig. 2 illustrates the Voronoi diagram of an example set of line segments.

When we are interested in the diagram $\mathsf{VD}(S)$ inside a compact region $A \subseteq \mathbb{R}^2$, we shall write $\mathsf{VD}_A(S)$ to denote the subdivision of A induced by $\mathsf{VD}(S)$. In other words, $\mathsf{VD}_A(S)$ is obtained by intersecting the diagram $\mathsf{VD}(S)$ itself with A.

The following fact is well known as the *star-shape* or *weak star-shape property* of Voronoi regions of a point or an open line segment.

Lemma 1. *If $s \in S$ is a point, then $R(s,S)$ is star-shaped with respect to s. If $s \in S$ is an open line segment, then $R(s,S)$ is weakly star-shaped in the sense that for any $x \in R(s,S)$, the segment $\overline{xs_x}$ is totally contained in $R(s,S)$, where s_x is the perpendicular foot from x to segment s.* $\square$