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Bundling three convex polygons to minimize area or perimeter ☆

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ABSTRACT

Given three convex polygons having n vertices in total in the plane, we consider the problem of finding a translation for each polygon such that the translated polygons are pairwise disjoint and the area or the perimeter of their convex hull is minimized. We present the first $O(n^2)$-time algorithm that finds optimal translations of input polygons using $O(n^2)$ space for this problem.

1. Introduction

The problem of bundling convex polygons in the plane under translation is to find a translation for each of them such that the translated polygons are contained in a smallest possible convex region while their interiors are disjoint. This problem can be modeled as follows: given a set $\mathcal{P} = \{P_0, \ldots, P_{k-1}\}$ of k convex polygons in the plane with n vertices in total, find k translations $\tau_0, \ldots, \tau_{k-1}$ of $P_0, \ldots, P_{k-1}$ such that the translated copies $\tau_i P_i$'s, for $0 \leqslant i \leqslant k-1$, do not overlap each other and the area or the perimeter of the convex hull of $\bigcup_{i=0}^{k-1} \tau_i P_i$ is minimized.

A problem closely related to the bundling problem is the *packing problem* of finding a smallest region, called a *container*, of a given shape (such as a disk, a square, or a rectangle) that packs the input objects under translations. Packing problems have received significant attention in a number of disciplines. For instance, it goes back to Kepler's conjecture (1611) on sphere packing in three-dimensional Euclidean space [9]. Sugihara et al. [12] considered a related problem of minimizing the disk bundling a set of disks with applications to minimizing the sizes of holes through which sets of electric wires are to pass. They proposed an $O(n^4)$-time heuristic method that makes use of the Voronoi diagram of circles, where n is the number of disks. Milenkovic studied the packing of a set of polygons into another polygon container with applications in the apparel industry [11]. He gave an $O(n^{k-1} \log n)$ time algorithm for packing k convex n-gons under translations into a minimum area axis-parallel rectangle container. Later, Alt and Hurtado [4] presented a near-linear time algorithm that packs two convex polygons into a minimum area or perimeter rectangle. Recently, Egeblad et al. [7] presented an efficient method for packing polytopes into another polytope container under translation in arbitrary dimension.

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Much less is known about the case when the container has no restriction on its shape. For $k = 2$, Lee and Woo [10] presented a linear time algorithm for finding a translation τ_1 of P_1 that minimizes the area of the convex hull of $P_0 \cup \tau_1 P_1$. Their algorithm can be easily extended to the case in which the perimeter of the convex hull is minimized.

It is well known that for two polygons P and Q, not necessarily convex, the Minkowski sum $P \oplus -Q$ of one and the negated copy of the other (also known as the *Minkowski difference*) describes the space of translations τ of Q such that P and τQ intersect each other [6]. More precisely, observe that for some $q \in Q$, $q + \tau \in P$ if and only if $\tau = p - q$ for some $p \in P$, equivalently, $\tau \in P \oplus -Q$. It is, however, not immediate how to extend the idea for the case of more than two polygons.

When k is part of input, the problem is shown to be NP-hard even if the polygons are rectangles [5], which was done by reducing the partition problem [8] into this problem.

There are also a couple of results that allow rigid motions of polygons for $k = 2$. Tang et al. [13] gave an $O(n^3)$-time algorithm for finding a rigid motion that minimizes the area of the convex hull of two convex polygons, where n is the total number of vertices. Ahn and Cheong [2,3] presented a near-linear time approximation algorithm for finding a rigid motion that minimizes either the perimeter or the area of the convex hull of two convex polygons.

Research extended into three or higher dimensional space. Ahn et al. [1] considered the problem of minimizing the volume or the surface area of the convex hull of two convex d-polytopes P and Q under translations without overlap for dimension $d \geqslant 3$. They presented an algorithm that computes an optimal translation in $O(n^{\lfloor \frac{d}{2} \rfloor (d-3)+d})$ time.

Our results We study the bundling problem in the plane when $k = 3$. Without loss of generality, we assume that P_0 is stationary. We show that the translation space of P_1 and P_2 can be decomposed into $O(n^2)$ cells in each of which the combinatorial structure of the convex hull remains the same. Moreover, we show that the description of the objective function for each cell can be fully described using constant space and the function description for a neighboring cell can be updated in constant time by coherence. This was done by a careful analysis on all event configurations at which the combinatorial structure of the convex hull changes. We then present an $O(n^2)$-time algorithm using $O(n^2)$ space that returns an optimal pair of translations.

2. Preliminaries

Let $P_0, \ldots, P_{k-1}$ be k convex polygons in $\mathbb{R}^2$ with n vertices in total. For a vector $\tau \in \mathbb{R}^{2k}$, we write $\tau = (\tau_0, \ldots, \tau_{k-1})$, where $\tau_i \in \mathbb{R}^2$. The *translate* of P_i by τ_i, denoted by $\tau_i P_i$, is $\{a + \tau_i \mid a \in P_i\}$. We let $U(\tau) = \bigcup_{i=0}^{k-1} \tau_i P_i$ and let $\mathrm{conv}(\tau) := \mathrm{conv}(U(\tau))$. The perimeter and the area of $\mathrm{conv}(\tau)$ are denoted by $|\mathrm{conv}(\tau)|$ and $\|\mathrm{conv}(\tau)\|$, respectively.

Ahn and Cheong [3] studied the area and perimeter functions and observed the following.

Lemma 1. *(See Ahn and Cheong [3].) The function $f : \mathbb{R}^{2k} \to \mathbb{R}$ with $f(\tau) = |\mathrm{conv}(\tau)|$ is convex for any $k \geqslant 2$. The function $g : \mathbb{R}^{2k} \to \mathbb{R}$ with $g(\tau) = \|\mathrm{conv}(\tau)\|$ is convex and piecewise linear for $k = 2$, but this is not necessarily the case for $k > 2$.*

The bundling problem can be viewed as an optimization problem of minimizing $|\mathrm{conv}(\tau)|$ or $\|\mathrm{conv}(\tau)\|$ over $\tau \in \mathbb{R}^{2k}$ subject to $\tau_i P_i \cap \tau_j P_j = \emptyset$ for all $0 \leqslant i < j \leqslant k - 1$. One can reduce the search space by a simple observation.

Lemma 2. *For the bundling problem with respect to either area or perimeter, there is an optimal translation vector $\tau^* \in \mathbb{R}^{2k}$ such that the union $U(\tau^*)$ is connected.*

Proof. If $U(\tau^*)$ is connected, we are done. Suppose that $U(\tau^*)$ consists of more than one connected component. Since any two connected components of $U(\tau^*)$ are disjoint in their closures, their convex hulls may overlap but do not cross.

If there is one component K such that $\mathrm{conv}(K) = \mathrm{conv}(\tau^*)$, then all the other components are contained in $\mathrm{conv}(K)$. We can always translate one of them to touch K and become connected to K keeping it contained in $\mathrm{conv}(K)$. This translation process makes no change to $\mathrm{conv}(\tau^*)$ but decreases the number of connected components by one. We repeat this process and finally have a single connected component.

Otherwise, there are at least two connected components whose convex hulls appear on the boundary of $\mathrm{conv}(\tau^*)$. Let L be the set of all such connected components of $U(\tau^*)$. Since the convex hulls of any two components of L do not cross, there is at least one connected component $K \in L$ whose convex hull appears on the boundary of $\mathrm{conv}(\tau^*)$ only once. Let e and e' be the two edges of $\mathrm{conv}(\tau^*)$ between K and the other components $\mathrm{conv}(L \setminus \{K\})$. Then one can translate K in the direction parallel to one of e and e' to have less area and perimeter, which contradicts to the optimality of τ^*. □

We can thus concentrate only on the cases where the k polygons are connected. We shall call $\tau \in \mathbb{R}^{2k}$ a *configuration* if $U(\tau)$ is connected. A configuration τ is *feasible* if and only if the interiors of the translates are disjoint under τ. Thus, our goal is to find an optimal feasible configuration with respect to area or perimeter.

Let $\mathcal{K}$ be the set of configurations for given k polygons $P_0, \ldots, P_{k-1}$. Each configuration $\tau \in \mathcal{K}$ is associated with several properties describing the structure of the convex hull $\mathrm{conv}(\tau)$. If $\tau_i P_i$ and $\tau_j P_j$ are in contact, then a vertex v of P_i lies