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On the edge crossing properties of Euclidean minimum weight Laman graphs ☆

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ABSTRACT

This paper is concerned with the crossing number of Euclidean minimum-weight Laman graphs in the plane. We first investigate the relation between the Euclidean minimum-weight Laman graph and proximity graphs, and then we show that the Euclidean minimum-weight Laman graph is quasi-planar and 6-planar. Thus the crossing number of the Euclidean minimum-weight Laman graph is linear in the number of points.

1. Introduction

A graph G is called *Laman* if $|E(G)| = 2|V(G)| - 3$ and $|E(H)| \leq 2|V(H)| - 3$ for any subgraph H of G with $E(H) \neq \emptyset$. A Laman graph has a property of being *minimally rigid* in the plane if it is realized as a *generic bar-joint framework* [11,7]. A bar-joint framework is a straight-line realization of a graph in the plane, and by regarding each edge as a bar and each point as a joint the rigidity of such a graph can be defined in a natural way (see, e.g., [7]). One of the most fundamental results in combinatorial rigidity theory asserts that a graph G realized on a generic point set (i.e., the set of the coordinates is algebraically independent over the rational field) is rigid if and only if G contains a spanning Laman subgraph [11]. Laman graphs appear in a wide range of applications, not only statics but also mechanical design such as linkages, design of CAD systems, analysis of protein flexibility, and sensor network localization [17,16].

Throughout the paper, by *a graph on a point set P* we mean a graph drawn in the plane with straight-line edges and vertex set P. In this paper we shall consider a bar-joint framework as a straight-line drawing in the plane, and we shall analyze geometric properties of the *Euclidean minimum-weight Laman graph* $\mathsf{MLG}(P)$ on a planar point set P, that is, a Laman graph on P with the minimum total edge-length over all Laman graphs on P. Throughout this paper we assume that no

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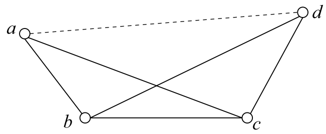

Fig. 1. MLG(P) that has an edge crossing.

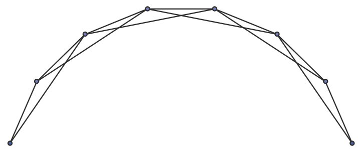

Fig. 2. MLG(P) that has $\Theta(|P|)$ edge crossings.

three points in P are collinear and all interpoint distances are distinct. The point set satisfying these assumptions is called *semi-generic* in this paper.

Our study is motivated by the well-known property of the Euclidean minimum spanning tree. For any semi-generic point set P on the plane, the Euclidean minimum spanning tree on P (MST(P) for short) is plane, where a graph on a point set is called *plane* (or *non-crossing*) if two edges do not have an intersection except possibly at their endpoints.

Observe that both Laman graphs and spanning trees are characterized by similar *sparsity conditions*. In general, a graph G is called (k,l)*-sparse* if $|E(H)| \leq k|V(H)| - l$ for any subgraph H of G with $E(H) \neq \emptyset$, and a (k,l)-sparse graph is called (k,l)*-tight* if it has exactly $k|V(G)| - l$ edges (see, e.g., [12]). A spanning tree is a $(1,1)$-tight graph while a Laman graph is a $(2,3)$-tight graph. (k,l)-sparse graphs have several common combinatorial properties such as being independent sets of a matroid. Hence a natural question is whether the Euclidean minimum-weight (k,l)-tight graph on a point set has a nice planarity property as does the Euclidean minimum-weight $(1,1)$-tight graphs in the plane. However it turns out that, unlike MST(P), MLG(P) may have a crossing in general (see Fig. 1) and there is a point set P for which MLG(P) has $\Theta(|P|)$ crossings as shown in Fig. 2.

One can describe the relation between MST(P) and the Delaunay triangulation in a detailed way by introducing *the nearest neighbor graph*, *the relative neighborhood graph*, *the Gabriel graph* and *the Delaunay graph* [15]. To define these graphs, we introduce some notation, which will be used throughout the paper. For two points $p, q \in \mathbb{R}^2$, $\|pq\|$ denotes the Euclidean distance between p and q. We abuse the notation pq to stand for $\|pq\|$ when there is no confusion. In particular, we write $pq < rs$ if the length of segment pq is less than that of rs. The closed disk with the segment pq as diameter is denoted by D_{pq}. Also, for a point $p \in \mathbb{R}^2$ and $r \in \mathbb{R}$, the closed disk (resp. circle) with center p and radius r is denoted by $D_p(r)$ (resp. $C_p(r)$).

In the $(k+1)$*-nearest neighbor graph* $(k+1)$-NNG(P), an edge pq is included if and only if p is the i-th closest point among P from q for some $i \leq k+1$ or vice versa. In the k*-relative neighborhood graph* k-RNG(P), pq is included if and only if $D_p(pq) \cap D_q(pq)$ contains at most k points among $P \setminus \{p,q\}$. In the k*-Gabriel graph* k-GG(P), pq is included if and only if D_{pq} contains at most k points among $P \setminus \{p,q\}$. In the k*-Delaunay graph* k-DG(P), pq is included if and only if there is a circle through p and q that contains at most k other points. As is well-known, 0-DG(P) is always a triangulation, called the Delaunay triangulation, if no four points lie on a circle. The following relations are classical (see for example [4,1,15]):

$$(k+1)\text{-NNG}(P) \subseteq k\text{-RNG}(P) \subseteq k\text{-GG}(P) \subseteq k\text{-DG}(P)$$

$$1\text{-NNG}(P) \subseteq \text{MST}(P) \subseteq 0\text{-RNG}(P).$$

In this context, we prove the next relations. (The proof is given at the end of Section 2.)

Theorem 1.1. *Let P be a semi-generic set of points in the plane. Then*

$$\text{MST}(P) \cup 2\text{-NNG}(P) \subseteq \text{MLG}(P) \subseteq 1\text{-GG}(P) \cap 2\text{-RNG}(P).$$

Ábrego et al. [1] recently investigated the crossing number and the maximum crossing number of proximity graphs, and they proved that k-NNG(P) has at most k^3n crossings for any P while there is a point set P such that k-GG(P) has $k^2n^2/4 + o(k^2n^2)$ crossings if $k = o(n)$. This result and Theorem 1.1 give rise to the following question: Does MLG(P) contain a linear number of crossings for every point set P? Our main theorem, proved in Section 4, affirmatively answers this question.

Theorem 1.2 *(6-Planarity theorem). Let P be a semi-generic set of points in the plane. For every edge $ab \in$ MLG(P), the number of edges crossing ab is at most six.*

Moreover, we prove the following theorem in Section 3.