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A polynomial-time approximation scheme for the geometric unique coverage problem on unit squares

Takehiro Ito [a], Shin-ichi Nakano [b], Yoshio Okamoto [c], Yota Otachi [d], Ryuhei Uehara [d], Takeaki Uno [e], Yushi Uno [f]

[a] Graduate School of Information Sciences, Tohoku University, Aoba-yama 6-6-05, Sendai 980-8579, Japan
[b] Department of Computer Science, Gunma University, Kiryu 376-8515, Japan
[c] Department of Communication Engineering and Informatics, University of Electro-Communications, Chofugaoka 1-5-1, Chofu, Tokyo 182-8585, Japan
[d] School of Information Science, Japan Advanced Institute of Science and Technology, Asahidai 1-1, Nomi, Ishikawa 923-1292, Japan
[e] Principles of Informatics Research Division, National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan
[f] Graduate School of Science, Osaka Prefecture University, 1-1 Gakuen-cho, Naka-ku, Sakai 599-8531, Japan

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ABSTRACT

We give a polynomial-time approximation scheme for the unique unit-square coverage problem: given a set of points and a set of axis-parallel unit squares, both in the plane, we wish to find a subset of squares that maximizes the number of points contained in exactly one square in the subset. Erlebach and van Leeuwen [9] introduced this problem as the geometric version of the unique coverage problem, and the best approximation ratio by van Leeuwen [21] before our work was 2. Our scheme can be generalized to the budgeted unique unit-square coverage problem, in which each point has a profit, each square has a cost, and we wish to maximize the total profit of the uniquely covered points under the condition that the total cost is at most a given bound.

1. Introduction

Let $\mathcal{P}$ be a set of points and $\mathcal{D}$ a set of axis-parallel unit squares,[1] both in the plane $\mathbb{R}^2$. For a subset $\mathcal{C} \subseteq \mathcal{D}$ of unit squares, we say that a point $p \in \mathcal{P}$ is *uniquely covered* by $\mathcal{C}$ if there is exactly one square in $\mathcal{C}$ containing p. In the *unique unit-square coverage problem*, we are given a pair $\langle \mathcal{P}, \mathcal{D} \rangle$ of a set $\mathcal{P}$ of points and a set $\mathcal{D}$ of axis-parallel unit squares as input, and we are asked to find a subset $\mathcal{C} \subseteq \mathcal{D}$ that maximizes the number of points uniquely covered by $\mathcal{C}$. An instance is shown in Fig. 1(a), and an optimal solution to this instance is illustrated in Fig. 1(b).

In a more general setting, in addition to an instance $\langle \mathcal{P}, \mathcal{D} \rangle$ of the unique unit-square coverage problem, we are given a non-negative real number B, called the *budget*, a non-negative real number $\mathsf{profit}(p)$ for each point $p \in \mathcal{P}$, called the *profit* of p, and a non-negative real number $\mathsf{cost}(S)$ for each square $S \in \mathcal{D}$, called the *cost* of S. In the *budgeted unique unit-square coverage problem*, we are asked to find a subset $\mathcal{C} \subseteq \mathcal{D}$ of total cost at most B such that the total profit of points in $\mathcal{P}$ uniquely covered by $\mathcal{C}$ is maximized. The unique unit-square coverage problem is a specialization of the budgeted unique unit-square coverage problem. To see this, set $\mathsf{profit}(p) = 1$ for all $p \in \mathcal{P}$, $\mathsf{cost}(S) = 0$ for all $S \in \mathcal{D}$, and $B = 0$.

E-mail addresses: takehiro@ecei.tohoku.ac.jp (T. Ito), nakano@cs.gunma-u.ac.jp (S.-i. Nakano), okamotoy@uec.ac.jp (Y. Okamoto), otachi@jaist.ac.jp (Y. Otachi), uehara@jaist.ac.jp (R. Uehara), uno@nii.ac.jp (T. Uno), uno@mi.s.osakafu-u.ac.jp (Y. Uno).

[1] Throughout this paper, a unit square is of side length one and is closed, thus contains the boundary.

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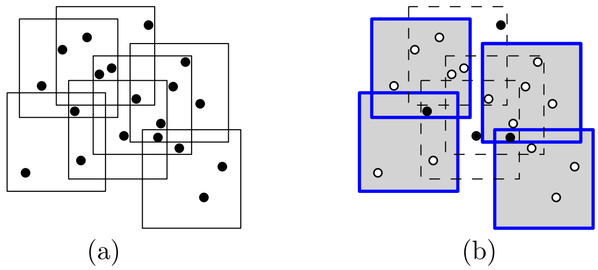

Fig. 1. (a) An instance $\langle \mathcal{P}, \mathcal{D} \rangle$ of the unique unit-square coverage problem and (b) an optimal solution to $\langle \mathcal{P}, \mathcal{D} \rangle$, where each square in the optimal solution is hatched and each uniquely covered point is drawn as a white circle.

1.1. Past work and motivation

Demaine et al. [7] formulated the non-geometric unique coverage problem in more general setting. They gave a polynomial-time $O(\log n)$-approximation algorithm[2] for the non-geometric unique coverage problem, where n is the number of elements (in the geometric version, n corresponds to the number of points). Guruswami and Trevisan [12] studied the same problem and its generalization, which they called the 1-in-k SAT. The unique coverage problem appears in several situations. The previous papers [7,12] provide a connection with unlimited-supply single-minded envy-free pricing and the maximum cut problem. For details, see their papers.

The parameterized complexity of the unique coverage problem has also been studied by Misra et al. [19].

Motivated by applications from wireless networks, Erlebach and van Leeuwen [9] studied the geometric versions of the unique coverage problem especially on unit disks. In the context of wireless networks, each point corresponds to a customer location, and the center of each disk corresponds to a place where the provider can build a base station. If several base stations cover a certain customer location, then the resulting interference might cause this customer to receive no service at all. Ideally, each customer should be serviced by exactly one base station. This situation corresponds to the unique unit-disk coverage problem. They showed that the problem on unit disks is strongly NP-hard, and gave a polynomial-time 18-approximation algorithm; for the budgeted unique unit-disk coverage problem, they provided a polynomial-time $(18 + \varepsilon)$-approximation algorithm for any fixed constant $\varepsilon > 0$ [9].

The unique unit-square coverage problem is an ℓ_∞ variant (or an ℓ_1 variant) of the unique unit-disk coverage problem. Erlebach and van Leeuwen [9] introduced the budgeted unique unit-square coverage problem, and gave a polynomial-time $(4 + \varepsilon)$-approximation algorithm for any fixed constant $\varepsilon > 0$. Later, van Leeuwen [21] gave a proof that the problem on unit squares is also strongly NP-hard, and improved the approximation ratio to $2 + \varepsilon$.

Optimization problems on axis-parallel unit squares and unit disks have been thoroughly studied since Huson and Sen [15]. A seminal paper by Hochbaum and Maass [13] established the shifting strategy, which has been used to give a polynomial-time approximation scheme (PTAS) for a lot of problems on unit squares and unit disks (see [14] for example). However, some problems such as coloring [6] and dispersion [11] (see also [8]) are APX-hard already for unit disks. The unique coverage problem is one among the problems for which we know the NP-hardness, but neither APX-hardness nor a PTAS was known. The existence of a PTAS for unit squares has been asked by van Leeuwen [21].

In a sister paper, we exhibit a polynomial-time approximation algorithm for the unique unit-disk coverage problem with approximation ratio $2 + 4/\sqrt{3} + \varepsilon$ ($< 4.3095 + \varepsilon$), where $\varepsilon > 0$ is any fixed constant [16].

After the conference version [17] of this paper was published, Chan and Hu [4] gave another PTAS for the unique unit-square coverage problem, which is, as they claim, "simpler to describe" than ours.

1.2. Contribution of the paper

In this paper, we give the first PTAS for the unique unit-square coverage problem, and hence we improve the approximation ratio to $1 + \varepsilon$ for any fixed constant $\varepsilon > 0$. The algorithm is generalized to give a PTAS for the budgeted unique unit-square coverage problem, too.

We employ the well-known shifting strategy, developed by Baker [1] and applied to the geometric problems by Hochbaum and Maass [13]. Namely, we partition the whole plane into "ribbons" of height one, and delete the points in every $1 + \lceil 1/\varepsilon \rceil$ ribbons. Then, the instance is divided into several subinstances in which all points lie in a rectangle of height $\lceil 1/\varepsilon \rceil$. We compute optimal solutions to such subinstances, and take their union. The best among all choices of possible deletions will be a $(1 + \varepsilon)$-approximate solution. On the other hand, van Leeuwen [21] was only able to solve a subinstance in a rectangle of height one, and thus only gave a 2-approximation since he removed the points in every two

[2] For notational convenience, throughout the paper, we say that an algorithm for a maximization problem is α-*approximation* if it returns a solution with the objective value APX such that $\mathrm{OPT} \le \alpha \mathrm{APX}$, where OPT is the optimal objective value, and hence $\alpha \ge 1$.