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Convex blocking and partial orders on the plane

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ABSTRACT

Let $C = \{c_1, \dots, c_n\}$ be a collection of disjoint closed bounded convex sets in the plane. Suppose that one of them, say c_1, represents a valuable object we want to uncover, and we are allowed to pick a direction $\alpha \in [0, 2\pi)$ along which we can translate (remove) the elements of C, one at a time, while avoiding collisions. We study the problem of finding a direction α_0 such that the number of elements that have to be removed along α_0 before we can remove c_1 is minimized. We prove that if we have the sorted set $\mathcal{D}$ of directions defined by the tangents between pairs of elements of C, we can find α_0 in $O(n^2)$ time. We also discuss the problem of sorting $\mathcal{D}$ in $o(n^2 \log n)$ time.

1. Introduction

Consider a set $C = \{c_1, \dots, c_n\}$ of pairwise disjoint closed bounded convex sets. It is well known that the elements of C can be removed one at a time by translating them upwards while avoiding collisions with other elements of C; see [11,16]. For example, the elements of the set $C = \{c_1, \dots, c_9\}$ shown in Fig. 1(a) can be removed in the order $c_2, c_3, c_1, c_9, c_6, c_4, c_5, c_7, c_8$. Clearly this result is also valid if we remove the elements of C by translating them along any direction $\alpha \in [0, 2\pi)$.

Suppose that $c_1 \in C$ is a special object that we want to uncover, and that we are allowed to choose a direction $\alpha \in [0, 2\pi)$ along which we can remove the elements of C one at a time while avoiding collisions. We want to find the direction α_0 that minimizes the number of elements we need to remove before we reach c_1. For example, in Fig. 1(b), it is easy to see that if we remove the elements of C in the direction α_1, four elements of C have to be removed before c_1 is uncovered, while for α_2 we only need to remove two.

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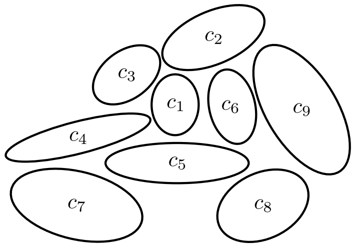
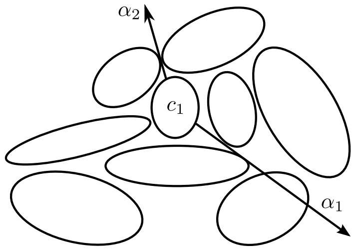


(a) The elements of C can be removed in the upward direction in the order $c_2, c_3, c_1, c_9, c_6, c_4, c_5, c_7, c_8$.



(b) Two different directions in which we can remove elements from C.

Fig. 1. Disassembly in different directions.

This problem can be seen as a variant of the problem known in computational geometry as the *separability problem* [5,4,9,17]. Similar problems are studied in [1,7], and it is also related to *spherical orders* determined by light obstructions [10].

In this paper, we present an $O(n^2)$ time algorithm to solve this problem, assuming that we have the sorted set $\mathcal{D}$ of directions defined by the tangents between pairs of elements of C. To ease our presentation, in the remainder of the paper we will assume that the interior of the convex sets is not empty. It is not hard to see that the result holds for families of closed sets.

In Section 2 we give basic definitions and state the problem in these terms. In Section 3 we explain how we can reduce the search space of our problem to the set $\mathcal{D}$ of critical directions. In Section 4 we present the data structure that we use to solve our problem. In Section 5, we present an algorithm to solve the main problem and we prove its time complexity. In Section 6, we discuss the difficulty of sorting $\mathcal{D}$ in less than $O(n^2 \log n)$ time. Lastly, in Section 7 we present our conclusions.

2. Partial orders and blocking

Let X be a finite set, and $<$ a relation on the elements of X that satisfies the following conditions:

1. If $x < y$ and $y < z$ then $x < z$ (transitivity), and
2. $x \not< x$ (anti-reflexivity).

The set X together with $<$ is called a partial order, and is usually denoted as $P(<, X)$.

Given $x, y \in X$, we say that y *covers* x if $x < y$ and there is no element $w \in X$ such that $x < w < y$. The *diagram* of $P(<, X)$ is the directed graph whose vertices are the elements of X, and which has an oriented edge from x to y if y covers x.

We say that the diagram of $P(<, X)$ is planar if it can be drawn on the plane in such a way that the following conditions are satisfied:

a) the elements of X are represented by points on the plane,
b) if y is a cover of x, the edge joining them is a monotonically increasing curve (with respect to the y-axis) starting at x and ending in y,
c) no edges of $P(<, X)$ intersect except perhaps at a common endpoint.

Given two elements $x, y \in X$, a *supremum* of x, y is an element $w \in X$ such that $x < w$, $y < w$, and for any other element $z \in X$ such that $x < z$ and $y < z$ we have that $w < z$. An *infimum* is defined in a similar way, except that we require w to be $w < x$ and $w < y$. An ordered set is called a *lattice* if any two elements have a unique supremum and infimum. A lattice is called a *planar lattice* if its diagram is planar. Finally, a partial order $P(<, X)$ is called a *truncated planar lattice* if the order that results when both a least and a greatest element are added to $P(<, X)$ is a planar lattice.

Let $C = \{c_1, \ldots, c_n\}$ be a set of disjoint closed bounded convex sets on the plane and $\alpha \in [0, 2\pi)$. Given two convex sets c_i and c_j in C, we say that c_j is an *upper cover* of c_i in the direction α (for short, an α-cover) if the following conditions are satisfied:

1. There is at least one *directed line segment* with direction α starting at a point in c_i and ending at a point in c_j.
2. Any directed line segment with direction α starting at a point in c_i and ending at a point in c_j does not intersect any other element of C.

Clearly, if c_j is an α-cover of c_i, then to uncover c_i along the α direction we need first to remove c_j. Observe that if c_j is an α-cover of c_i, then c_i is an $(\alpha + \pi)$-cover of c_j. We say that c_j *blocks* c_i in the direction α, written as $c_i \prec_\alpha c_j$, if there is a sequence $c_i = c_{\sigma(1)}, c_{\sigma(2)}, \ldots, c_{\sigma(k)} = c_j$ of elements of C such that $c_{\sigma(r+1)}$ is an α-cover of $c_{\sigma(r)}$, with $r = 1, \ldots, k-1$ (Fig. 2). The following observation will be useful: