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Drawing outerplanar graphs using three edge lengths

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ABSTRACT

It is shown that for any outerplanar graph G there is a one to one mapping of the vertices of G to the plane, so that the number of distinct distances between pairs of connected vertices is at most three. This settles a problem of Carmi, Dujmović, Morin and Wood. The proof combines (elementary) geometric, combinatorial, algebraic and probabilistic arguments.

1. Introduction

A *linear* embedding of a graph G is a mapping of the vertices of G to points in the plane. The image of every edge uv of the graph is the open interval between the image of u and the image of v. The length of that interval is called the *edge-length* of uv in the embedding. A *degenerate drawing* of a graph G is a linear embedding in which the images of all vertices are distinct. A *drawing* of G is a degenerate drawing in which the image of every edge is disjoint from the image of every vertex. The *distance-number* of a graph is the minimum number of distinct edge-lengths in a drawing of G, the *degenerate distance-number* is its counterpart for degenerate drawings.

An *outerplanar* graph is a graph that can be embedded in the plane without crossings in such a way that all the vertices lie in the boundary of the unbounded face of the embedding. In [1], Carmi, Dujmović, Morin and Wood ask if the degenerate distance-number of outerplanar graphs is uniformly bounded. We answer this positively by showing that the degenerate distance number of outerplanar graphs is at most 3. This result is derived by explicitly constructing a degenerate drawing for every such graph.

Theorem 1. *For almost every triple $a, b, c \in (0, 1)$, every outerplanar graph has a degenerate drawing using only edge-lengths a, b and c.*

For matters of convenience, throughout the paper we consider all linear embeddings as mapping vertices to the complex plane.

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2. Background and motivation

While the distance-number and the degenerate distance-number of a graph are two natural notions in the context of representing a graph as a diagram in the plane, this was not the sole motivation to their introduction.

Both notions were introduced by Carmi, Dujmovic, Morin and Wood in [1], and generalize several well studied problems. Indeed, Erdős suggested in [2] the problem of determining or estimating the minimum possible number of distinct distances between n points in the plane. This problem can be rephrased as finding the degenerate distance-number of K_n, the complete graph on n vertices. Recently, Guth and Katz, in a ground-breaking paper [3], established a lower-bound of $cn/\log n$ on this number, which almost matches the $O(n/\sqrt{\log n})$ upper-bound due to Erdős. Another problem, considered by Szemerédi (see Theorem 13.7 in [5]), is that of finding the minimum possible number of distances between n non-collinear points in the plane. This problem can be rephrased as finding the distance-number of K_n. One interesting consequence of the known results on these questions is that the distance-number and the degenerate distance-number of K_n are not the same, thus justifying the two separate notions. For a short survey of the history of both problems, including some classical bounds, the reader is referred to the background section of [1].

Another notion which is generalized by the degenerate distance-number is that of a unit-distance graph, that is, a graph that can be embedded in the plane so that two vertices are at distance one if and only if they are connected by an edge. Observe that all unit-distance graphs have degenerate distance-number 1 while the converse is not true. This is because in a degenerate drawing of a graph which uses only one edge length, there could be points at distance 1 which are not connected by an edge. Constructing "dense" unit-distance graphs is a classical problem. The best construction, due to Erdős [2], gives an n-vertex unit-distance graph with $n^{1+c/\log\log n}$ edges, while the best known upper-bound, due to Spencer, Szemerédi and Trotter [6], is $cn^{4/3}$ (a simpler proof for this bound was found by Székely, see [7]). Note that this implies that the k most frequent interpoint distances between n points occur in total no more than $ckn^{4/3}$ times, and thus that a graph with degenerate distance-number k cannot have more than $ckn^{4/3}$ edges. Katz and Tardos gave in [4] another bound on the frequency of interpoint distances between n points in the plane, which yields that a graph with distance-number k cannot have more than $cn^{1.46}k^{0.63}$ edges.

After introducing the notions of distance-number and degenerate distance-number, Carmi, Dujmovic, Morin and Wood studied in [1] the behavior of bounded degree graphs with respect to these notions. They show that graphs with bounded degree greater or equal to five can have degenerate distance-number arbitrarily large, giving a polynomial lower-bound for graphs with bounded degree greater or equal to seven. They also give a $c\log(n)$ upper-bound to the distance-number of bounded degree graphs with bounded treewidth. In the same paper, the authors ask whether this bound can be improved for outerplanar graphs, and in particular whether such graphs have a uniformly bounded degenerate distance-number, a question which we answer here positively.

3. Preliminaries

Outerplanarity, Δ-trees and T^*. An *outerplanar* graph is a graph that can be embedded in the plane without crossings so that all its vertices lie in the boundary of the unbounded face of the embedding. The edges which border this unbounded face are uniquely defined, and are called the *external* edges of the graph; the rest of the edges are called *internal*.

Let Δ be the triangle graph, that is, a graph on three vertices v_0, v_1, and v_2, whose edges are v_0v_1, v_0v_2 and v_2v_1. A graph is said to be a *Δ-tree* if it can be generated from Δ by iterations of adding a new vertex and connecting it to both ends of some external edge other than v_0v_1. This results in an outerplanar graph whose bounded faces are all triangles. The adjacency graph of the bounded faces of such a graph is a binary tree, that is – a rooted tree of maximal degree 3. In fact, all Δ-trees are subgraphs of an infinite graph T^*. All bounded faces of T^* are triangles, and the adjacency graph of those faces is a complete infinite binary tree. The root of T^* is denoted by T^*_{root}. An illustration of a Δ-tree can be found on the left hand side of Fig. 3.

It is a known fact, which can be proved using induction, that the triangulation of every outerplanar graph is a Δ-tree. All outerplanar graphs are therefore subgraphs of T^*, a fact which reduces Theorem 1 to the following:

Proposition 1. *For almost every triple $a, b, c \in (0, 1)$, the graph T^* has a degenerate drawing using only edge-lengths a, b and c.*

The rhombus graph H, covering T^* by rhombi. In order to prove the above proposition, we construct an explicit embedding of T^* in $\mathbb{C}$. To do so we introduce a covering of T^* by copies of a particular directed graph H which we call a *rhombus*. We then embed T^* into $\mathbb{C}$, one copy of H at a time.

The *rhombus* directed graph H (see Fig. 1), is defined to be the graph satisfying $V_H = \{v_0, v_1, v_2, v_3\}$ and $E_H = \{v_0v_1, v_0v_2, v_2v_3, v_1v_3, v_2v_1\}$. We call v_0 the *base vertex* of H.

We further define H^* to be the infinite directed trinary tree whose nodes are copies of H, labeling the three arcs emanating from every node by v_0v_2, v_2v_3 and v_1v_3. We write $L(a)$ for the label of an arc a. Let N be a node of H^*, and let $v_iv_j \in E_H$; we call a pair (N, v_i) a vertex of H^*, and a pair (N, v_iv_j), an edge of H^*. Notice the distinction between arcs of H^* and edges of H^*, and the distinction between nodes and vertices. The root of H^* is denoted by H^*_{root}. A portion of H^* is depicted in Fig. 2.