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Flip distance between two triangulations of a point set is NP-complete

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Given two triangulations of a convex polygon, computing the minimum number of flips required to transform one to the other is a long-standing open problem. It is not known whether the problem is in P or NP-complete. We prove that two natural generalizations of the problem are NP-complete, namely computing the minimum number of flips between two triangulations of (1) a polygon with holes; (2) a set of points in the plane.

1. Introduction

Given a triangulation in the plane, a *flip* operates on two triangles that share an edge and form a convex quadrilateral. The flip replaces the diagonal of the convex quadrilateral by the other diagonal to form two new triangles. A sequence of flips can transform any triangulation to any other triangulation—this is true for triangulations of a convex polygon, and more generally for triangulations of a point set and for triangulations of a polygon with holes.

In this paper we investigate the complexity of computing the *flip distance*, which is the minimum number of flips to transform one triangulation to another. This is particularly interesting for convex polygons, where the flip distance is the rotation distance between two binary trees (see below).

The main result of our paper is that it is NP-complete to compute the flip distance between two triangulations of a polygon with holes, or of a set of points in the plane.

After submitting this paper, we learned that Pilz [23] independently proved the same result, and then strengthened it to prove APX-hardness. The differences between our proofs are discussed later on.

1.1. Flip distance and rotation distance

Binary search trees are a widely used data structure, and *rotations* are the basic operations used to balance them. Despite the importance of rotations, the complexity of computing the minimum number of rotations to convert one labeled binary search tree to another, called the “rotation-distance”, has been open since at least 1982 [8]. It is not known if the problem is NP-complete.

There is a bijection between binary trees with $n-1$ labeled leaves and triangulations of an n-vertex convex polygon. Moreover, a rotation in the tree corresponds to a flip in the polygon. Thus, computing the rotation distance between two trees is exactly equivalent to computing the flip distance between two triangulations of a convex polygon. See [25].

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1.2. Generalizations and related work

Flips have been studied in the geometric setting for triangulations of point sets and of polygons. In this context, a convex polygon is equivalent to a point set in convex position. The former generalizes to simple polygons, and the latter to planar point sets. Both of these are contained in the most general case of a polygon with holes (a "polygonal region"), so long as we consider a point as a one vertex polygonal hole. There is a survey on flips by Bose and Hurtado [6]. It also covers flips in the combinatorial setting of maximal planar graphs, which we will not discuss. Flips are often studied in terms of the *flip graph* which has a vertex for every triangulation and an edge when two triangulations differ by one flip, see e.g., [13].

The foundational result is that the flip graph is connected. This was proved first by Lawson [17] for the case of point sets. He then re-proved the result [16] by arguing that any triangulation can be flipped to the Delaunay triangulation, which then acts as a "canonical" triangulation from which any other triangulation can be reached. The constrained Delaunay triangulation can be used in the same way to argue that any polygonal region has a connected flip graph [4]. For more direct proofs see [12,15,21].

Regarding the number of flips needed to transform one triangulation to another, flipping via the [constrained] Delaunay triangulation takes $O(n^2)$ flips—in fact, a more exact bound is the number of visibility edges, see [4]. Hurtado, Noy and Urrutia [15] proved that $\Omega(n^2)$ flips may be required even for triangulations of a polygon. For the case of a convex polygon, Sleator et al. [25] proved that for large values of n, the flip distance between two triangulations of an n-gon is at most $2n-10$, and that $2n-10$ flips are sometimes necessary.

The problem of computing the exact flip distance between two given triangulations is especially interesting for convex polygons, as mentioned above. Lucas [19] gave a polynomial time algorithm for special cases. The best approximation factor is trivially 2, and can be improved in some special cases [18]. Recently it was proved that the problem is fixed-parameter tractable in the flip distance [7]. Attempts have also been made to compute good upper and lower bounds on the flip distance efficiently. See, for example, [3,10,20,22].

The more general problem of computing the flip distance between two triangulations of a point set is stated as an open problem in the survey by Bose and Hurtado [6], the book by Devadoss and O'Rourke [11, Unsolved Problem 12] and the book on triangulations by De Loera et al. [9, Exercise 3.18]. Hanke et al. [14] proved that the flip-distance is upper bounded by the total number of intersections between the overlap of the initial and final triangulations. Eppstein [13] provided an algorithm to compute a lower bound on the flip-distance efficiently. He also showed that the lower bound is equal to the flip-distance for certain special kinds of point sets. In a recent work by Aichholzer et al. [1], the problem of computing the flip distance was also shown to be NP-complete for triangulations of a simple polygon.

2. Triangulations of polygonal regions

Theorem 1. *The following problem is NP-complete: Given two triangulations of a polygon with holes and a number k, is the flip distance between the two triangulations at most k?*

2.1. Proof idea

Note that the problem lies in NP since the flip-sequence of size at most k is itself a polynomial-sized certificate. We prove hardness by giving a polynomial time reduction from vertex cover on 3-connected cubic planar graphs [5,26], which is known to be NP-complete [5,26].

The idea is to take a planar straight-line drawing of the graph and create a polygonal region by replacing each edge by a "channel" and each vertex by a "vertex gadget". We then construct two triangulations of the polygonal region that differ on the channels, and show that a short flip sequence corresponds to a small vertex cover in the original graph.

We begin by describing channels and their triangulations, because this gives the intuition for the proof. A *channel* is a polygon that consists of two 7-vertex reflex chains joined by two *end* edges, as shown in Figs. 1(a) and 1(b). Note that every vertex on the upper reflex chain sees every vertex on the lower reflex chain and vice versa. We identify two triangulations of a channel: a *left-inclined triangulation* as shown in Fig. 1(a); and a *right-inclined triangulation* as shown in Fig. 1(b).

A channel is the special case $n=7$ of the polygons H_n of Hurtado et al. [15]. They prove in Theorem 3.8 that the flip distance between the right-inclined and left-inclined triangulations of H_n is $(n-1)^2$. We include a different proof in order to generalize:

Property 1. *Transforming a left-inclined triangulation of a channel to a right-inclined triangulation takes at least 36 flips.*

Proof. In any triangulation of a channel, each edge of the upper reflex chain is in a triangle whose apex lies on the bottom reflex chain. This apex must move from lower right (B_7) to lower left (B_1), in order to transform the left-inclined triangulation to the right-inclined triangulation. Similarly, each edge of the lower reflex chain is in a triangle whose apex lies on the upper reflex chain, and must move from upper left to upper right. However, one flip can only involve one edge of the upper chain and one edge of the lower chain (no other 4 vertices form a convex quadrilateral), and thus can only move one upper and one lower apex, and only by one vertex along the chain. Twelve triangles times six apex moves per triangle divided by two apex moves per flip gives a lower bound of 36 flips. □