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On grids in topological graphs

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ABSTRACT

A *topological graph* G is a graph drawn in the plane with vertices represented by points and edges represented by continuous arcs connecting the vertices. If every edge is drawn as a straight-line segment, then G is called a *geometric graph*. A k*-grid* in a topological graph is a pair of subsets of the edge set, each of size k, such that every edge in one subset crosses every edge in the other subset. It is known that every n-vertex topological graph with no k-grid has $O_k(n)$ edges.

We conjecture that the number of edges of every n-vertex topological graph with no k-grid such that all of its $2k$ edges have distinct endpoints is $O_k(n)$. This conjecture is shown to be true apart from an iterated logarithmic factor $\log^* n$. A k-grid is *natural* if its edges have distinct endpoints, and the arcs representing each of its edge subsets are pairwise disjoint. We also conjecture that every n-vertex geometric graph with no natural k-grid has $O_k(n)$ edges, but we can establish only an $O_k(n\log^2 n)$ upper bound. We verify the above conjectures in several special cases.

1. Introduction

The *intersection graph* of a set $\mathcal{C}$ of geometric objects has vertex set $\mathcal{C}$ and two objects are connected by an edge if and only if their intersection is nonempty. The problems of finding a maximum independent set and a maximum clique in the intersection graph of geometric objects have received considerable attention in the literature due to their applications in VLSI design [9], map labeling [1], frequency assignment in cellular networks [12], and elsewhere. Here we study the intersection graph of the edge set of graphs that are drawn in the plane. It is known that if such an intersection graph does not contain a large complete bipartite subgraph, then the number of edges in the original graph is small [7,15]. We show that this statement remains true under much weaker assumptions.

A *topological graph* G is a graph drawn in the plane with points as vertices and edges as Jordan arcs between these vertices. We further assume that (1) no arc passes through any vertex different from its endpoints, (2) every pair of edges

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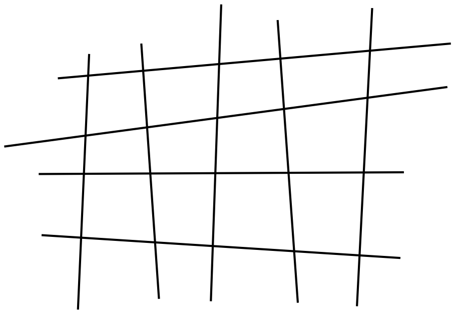

Fig. 1. A "natural" grid.

have only finitely many interior points in common, and (3) at each of these points the two edges properly cross. We only consider graphs without parallel edges or self-loops. A topological graph is *simple* if every pair of its edges intersect in at most one point, which is either a common endpoint or a proper crossing. If the edges are drawn as straight-line segments, then G is called a *geometric graph*.

Given a topological graph G, the intersection graph of its edge set is an abstract graph with vertex set $E(G)$, where two elements of $E(G)$ are connected by an edge if and only if they cross each other (a common endpoint of two edges does not count as a crossing). A complete bipartite subgraph in the intersection graph of $E(G)$ corresponds to a *grid* structure in G.

Definition 1.1. A (k,l)-*grid* in a topological graph is a pair of edge subsets E_1, E_2 such that $|E_1| = k$, $|E_2| = l$, and every edge in E_1 crosses every edge in E_2. A *k-grid* is an abbreviation for a (k,k)-grid.

Theorem 1.2. *(See [15].) For given integers $k, l \geqslant 1$, there exists a constant $c_{k,l}$ such that any topological graph on n vertices with no (k,l)-grid has at most $c_{k,l}n$ edges.*

The proof of Theorem 1.2 in [15] actually guarantees that a graph with many edges must contain a grid in which all the edges of one of the subsets are adjacent to a common vertex. For two recent and different proofs of Theorem 1.2, see [8] and [7]. Tardos and Tóth [20] extended the result in [15] by showing that there is a constant c_k such that a topological graph on n vertices and at least $c_k n$ edges must contain three subsets of k edges each, such that every pair of edges from different subsets cross, and for two of the subsets all the edges within the subset are adjacent to a common vertex.

Note that, according to Definition 1.1, the edges within each subset of the grid are allowed to cross or share a common vertex, as is indeed required in the proofs of [15] and [20]. When we thinks of a "grid", usually a drawing similar to Fig. 1 comes to our mind. Here the edges participating in the grid form a matching and each of the two edge sets consists of disjoint edges. More precisely, we define a *natural* (k,l)-*grid* in a topological graph G, as a pair of subsets $E_1, E_2 \subset E(G)$ with $|E_1| = k$, $|E_2| = l$ such that all $2(k+l)$ endpoints of $E_1 \cup E_2$ are distinct, the edges in E_1 are pairwise disjoint, and the edges in E_2 are pairwise disjoint.

Conjecture 1.3. *For given integers $k, l \geqslant 1$ there exists a constant $c_{k,l}$, such that any simple topological graph G on n vertices with no natural (k,l)-grid has at most $c_{k,l}n$ edges.*

Note that it is already not easy to show that an n-vertex geometric graph with no k pairwise disjoint edges has $O_k(n)$ edges (see [17] and [21]). Moreover, it is an open question whether a simple topological graph on n vertices and no k disjoint edges has $O_k(n)$ edges (the best upper bound, due to Pach and Tóth [16], is $O_k(n \log^{4k-8} n)$). Therefore, probably it is not an easy task to prove Conjecture 1.3. Here we establish the following upper bounds for the number of edges of geometric and simple topological graphs that contain no natural k-grids.

Theorem 1.4.

(i) *Every n-vertex geometric graph with no natural k-grid has $O(k^2 n \log^2 n)$ edges.*
(ii) *Every n-vertex simple topological graph with no natural k-grid has $O_k(n \log^{4k-6} n)$ edges.*

We phrased Conjecture 1.3 only for *simple* topological graphs, because it is false without this assumption. Indeed, one can draw the complete graph as a topological graph so that every pair of its edges intersect. (It can be done even in such a way that every pair of edges cross at most twice [16].) Therefore, for topological graphs we have to strengthen the condition by excluding all grids with distinct endpoints.