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Computing homotopic line simplification

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ABSTRACT

In this paper, we study a variant of the well-known line-simplification problem. For this problem, we are given a polygonal path $\mathcal{P} = p_1, p_2, \ldots, p_n$ and a set $\mathcal{S}$ of m point obstacles in the plane, with the goal being to determine an optimal homotopic simplification of $\mathcal{P}$. This means finding a minimum subsequence $\mathcal{Q} = q_1, q_2, \ldots, q_k$ ($q_1 = p_1$ and $q_k = p_n$) of $\mathcal{P}$ that approximates $\mathcal{P}$ within a given error ε that is also homotopic to $\mathcal{P}$. In this context, the error is defined under a distance function that can be a Hausdorff or Fréchet distance function, sometimes referred to as the error measure. In this paper, we present the first polynomial-time algorithm that computes an optimal homotopic simplification of $\mathcal{P}$ in $O(n^6m^2) + T_F(n)$ time, where $T_F(n)$ is the time to compute all shortcuts p_ip_j with errors of at most ε under the error measure F. Moreover, we define a new concept of strongly homotopic simplification where every link q_lq_{l+1} of $\mathcal{Q}$ corresponding to the shortcut p_ip_j of $\mathcal{P}$ is homotopic to the sub-path $p_i, \ldots, p_j$. We present a method that in $O(n(m+n)\log(n+m))$ time identifies all such shortcuts. If $\mathcal{P}$ is x-monotone, we show that this problem can be solved in $O(m\log(n+m) + n\log n\log(n+m) + k)$ time, where k is the number of such shortcuts. We can use Imai and Iri's framework [24] to obtain the simplification at the additional cost of $T_F(n)$.

1. Introduction

Motivation. Visualization of a large geographical map may require different levels of simplifications. A map may consist of a collection of non-intersecting chains representing features, such as rivers or country borders, and of points representing places, such as cities, etc. A simplified map of interest resembles the original map in the following aspects: (i) the distance between each point on the original chain and its corresponding simplified chain should be within a given error tolerance and (ii) the original chain and its simplified version must be in the same homotopy class.[1] Roughly speaking, this means that if a point (a city, for instance) lies below the original chain (a river, for example), it must also remain below the simplified chain. We, however, consider a simpler variant of the above simplification criteria, which will be described below.

Problem. We are given a polygonal path $\mathcal{P} = p_1, p_2, \ldots, p_n$, a set $\mathcal{S} = \{s_1, \ldots, s_m\}$ of m point obstacles in the plane not intersecting $\mathcal{P}$, and an error ε defined under a distance function F. The problem is to simplify the path $\mathcal{P}$ by $\mathcal{Q} = q_1, q_2, \ldots, q_k$ ($q_1 = p_1$ and $q_k = p_n$) within the given error ε so that $\mathcal{Q}$ is homotopic (to be exactly defined later) to $\mathcal{P}$.

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[1] A homotopy class is the class of all paths that are homotopic to each other in a plane.

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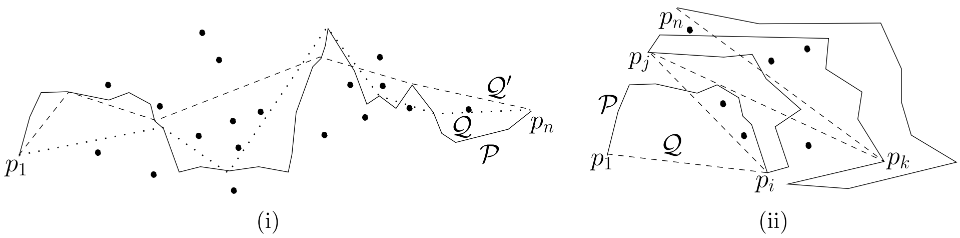

Fig. 1. (i) Two simplifications $\mathcal{Q}$ and $\mathcal{Q}'$ of $\mathcal{P}$. Only $\mathcal{Q}$ is homotopic to $\mathcal{P}$. (ii) While the simplified path $\mathcal{Q}$ and the original path $\mathcal{P}$ are homotopic, some shortcuts such as p_1p_i and p_jp_k are not homotopic to $\mathcal{P}(1,i)$ and $\mathcal{P}(j,k)$, respectively. Therefore, $\mathcal{P}$ and $\mathcal{Q}$ are not strongly homotopic.

Background. The above problem is a variant of the well-known line-simplification problem, also known in the literature as path, curve, or chain simplification, in which, for a given $\mathcal{P}$, the goal is to find the simplified path $\mathcal{Q}$ with fewer vertices approximating $\mathcal{P}$ within ε. This problem arises in many applications, such as GIS [5,6,15], image processing, and/or computer graphics [9,14], where reduction of the volume of data or lowering the complexity of the costly processing operations is important. In some of the applications, preserving the homotopy of the shape is also desirable. This ensures that the *aboveness relation*[2] of points and chains in the original and simplified maps remains unchanged.

Many variants of the line-simplification problems have been considered in the past, which can be classified into two main versions—*unrestricted* and *restricted*. In the former, the vertices of $\mathcal{Q}$ are allowed to be any arbitrary points, not just the vertices of $\mathcal{P}$ (see [20,21,23] for some results). In the restricted version, the vertices of $\mathcal{Q}$ are a subsequence of $\mathcal{P}$, and each segment q_lq_{l+1} is called a *link*. In this paper, we focus on the restricted version.

Each segment p_ip_j is called a shortcut, and each link q_lq_{l+1} of $\mathcal{Q}$ corresponds to a shortcut p_ip_j (with $j > i$). The error of such a link is defined as the distance between p_ip_j and the sub-path $p_i, \ldots, p_j$ (denoted by $\mathcal{P}(i,j)$) under a desired distance function F, which is often Hausdorff or Fréchet. The total *error* of $\mathcal{Q}$ denoted by $\text{error}(\mathcal{Q},\mathcal{P})$ is also defined as the maximum error among all of its links. For each distance function, there exist two constrained optimization problems: (i) min-#: considering that $\mathcal{P}$ and ϵ are given, compute $\mathcal{Q}$ with the minimum number of vertices in such a way that $\text{error}(\mathcal{Q},\mathcal{P}) \leqslant \epsilon$, and (ii) min-$\epsilon$: considering that $\mathcal{P}$ and a maximum number of vertices k are given, compute $\mathcal{Q}$ of $\mathcal{P}$ with the smallest possible error in a way that it uses at most k vertices. The min-ϵ version is usually computed by performing a binary search over the pre-computed errors and by applying a min-# algorithm at each step. In this paper, we focus on the min-# version in the restricted model. For brevity, we use "simplification" for "min-# line simplification in the restricted model" to avoid confusion.

The simplified path $\mathcal{Q}$ is homotopic to the original path $\mathcal{P}$ (or $\mathcal{Q}$ and $\mathcal{P}$ are in the same homotopy class) if it (the simplified path $\mathcal{Q}$) is continuously deformable to $\mathcal{P}$ without passing over any points of $\mathcal{S}$ while keeping its end-vertices fixed. Precisely, the two paths α and $\beta : [0,1] \to \mathbb{R}^2$, sharing the start and end points, are homotopically equivalent with respect to $\mathcal{S}$ if there exists a continuous function $\Gamma : [0,1] \times [0,1] \to \mathbb{R}^2$ with the following properties:

(i) $\Gamma(0,t) = \alpha(t)$ and $\Gamma(1,t) = \beta(t)$ for $0 \leqslant t \leqslant 1$,
(ii) $\Gamma(s,0) = \alpha(0) = \beta(0)$, and $\Gamma(s,1) = \alpha(1) = \beta(1)$, and
(iii) $\Gamma(s,t) \notin \mathcal{S}$ for $0 \leqslant s \leqslant 1$ and $0 < t < 1$.

Fig. 1(i) illustrates two simplifications of $\mathcal{P}$ where $\mathcal{Q}$ is homotopic to $\mathcal{P}$, but $\mathcal{Q}'$ is not. We define the concept of strongly homotopic as follows: $\mathcal{Q}$ is *strongly homotopic* to $\mathcal{P}$ if for any link q_lq_{l+1} of $\mathcal{Q}$ corresponding to the shortcut p_ip_j, the sub-path $\mathcal{P}(i,j)$ and p_ip_j are also homotopic. Such a shortcut is also called a *homotopic shortcut*. Obviously, if $\mathcal{Q}$ is strongly homotopic to $\mathcal{P}$, then they are homotopic to each other, but the reverse is not necessarily true, as shown in Fig. 1(ii). However, it is easy to conclude that any x-monotone chain is strongly homotopic to any of its homotopic simplifications. We can obviously think of applications for this concept, even though its theoretical impact is of more interest to us. For example, imagine a robot and a utility wagon inside a building that includes some pillars. The robots should move along a predefined polygonal path and fix some issues in the building using the utilities in the wagon. As the wagon movement is costly, we want to minimize this cost. Furthermore, the robot cannot access the wagon if it is not within the distance ε from it. To solve this problem, we find the strongly homotopic minimum path simplification under Fréchet distance. Note that the robot cannot rotate its hand to pick up a utility from the wagon if there is a pillar between them.

Related works. There are few results on line simplification in the presence of other objects that observe the concept of homotopy type. algorithm [15,22], which is a heuristic method and does not guarantee an optimal solution. Imai and Iri [24] solved the min-ε and the min-# versions by modeling each version as a shortest-path problem in the directed

[2] Object p (point, segment or x-monotone path) is above object q if there exists a vertical line intersecting both p and q, satisfying that p is above q with respect to this vertical line.