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Reprint of: Extreme point and halving edge search in abstract order types

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ABSTRACT

Many properties of finite point sets only depend on the relative position of the points, e.g., on the order type of the set. However, many fundamental algorithms in computational geometry rely on coordinate representations. This includes the straightforward algorithms for finding a halving line for a given planar point set, as well as finding a point on the convex hull, both in linear time. In his monograph *Axioms and Hulls*, Knuth asks whether these problems can be solved in linear time in a more abstract setting, given only the orientation of each point triple, i.e., the set's chirotope, as a source of information. We answer this question in the affirmative. More precisely, we can find a halving line through any given point, as well as the vertices of the convex hull edges that are intersected by the supporting line of any two given points of the set in linear time. We first give a proof for sets realizable in the Euclidean plane and then extend the result to non-realizable abstract order types.

1. Introduction

In computational geometry, many fundamental properties of finite point sets do not depend on the actual coordinates of each point in real space, but rather on the relative position of the points among each other. In their landmark paper, Goodman and Pollack [1] capture this idea by defining the order type of a point set. In the plane, two point sets have the same *order type* if there is a bijection π between the sets s.t. for every triple p, q, r of the first set, the corresponding points $\pi(p), \pi(q)$, and $\pi(r)$ have the same orientation (i.e., are both oriented clockwise or counterclockwise).[3] This orientation can be tested by the inequality

$$det\begin{pmatrix} p_x & p_y & 1 \\ q_x & q_y & 1 \\ r_x & r_y & 1 \end{pmatrix} > 0,$$

which indicates whether r is to the left of the directed line through p and q, i.e., whether the triple is oriented counterclockwise. The sign of the determinant therefore gives a predicate $\nabla(p, q, r)$ that is true iff the triple is oriented counterclockwise.

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[3] It is common to also consider sets to be of the same order type if the orientation of all triples is inverted in the second set, i.e., the second set can be seen as a mirrored copy of the first set.

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This mapping of all triples of a set to their orientation is also called the *chirotope* of the set (cf. Remark 1.6 in [2] and [3, p. 95] for details on that term). Many combinatorial properties of a set of points only depend on its order type, like its convex hull, the set of its crossing-free graphs, etc. We implicitly assume throughout this paper that all sets are in general position, i.e., do not contain collinear triples.

In contrast to these properties, there are further, more "metric" properties of a point set that are not determined by the order type. This includes the set's Delaunay triangulation; it is straightforward to construct two sets of the same order type whose Delaunay triangulations are different. Nevertheless, the problem can still be considered as being discrete. Guibas and Stolfi [4] separate topological from geometric aspects, using a predicate $\texttt{InCircle}(p, q, r, s)$ that is true iff the triple (p, q, r) is oriented counterclockwise and the point s lies inside the circle defined by the first three points. This predicate is equivalent to

$$det \begin{pmatrix} p_x & p_y & p_x^2 + p_y^2 & 1 \\ q_x & q_y & q_x^2 + q_y^2 & 1 \\ r_x & r_y & r_x^2 + r_y^2 & 1 \\ s_x & s_y & s_x^2 + s_y^2 & 1 \end{pmatrix} > 0.$$

Their Delaunay triangulation algorithm depends almost entirely on this predicate, making it a robust approach, that is intended to be easy to implement and to prove.

Motivated by this approach, Knuth [3] develops axiomatic systems following these two tests. He defines five axioms over a ternary predicate P and calls sets of triples obeying them *CC Systems*.

Axiom 1 *(cyclic symmetry).* $P(p, q, r) \Rightarrow P(r, p, q)$.

Axiom 2 *(antisymmetry).* $P(p, q, r) \Rightarrow \neg P(p, r, q)$.

Axiom 3 *(nondegeneracy).* $P(p, q, r) \vee P(p, r, q)$.

Axiom 4 *(interiority).* $P(t, p, q) \wedge P(t, q, r) \wedge P(t, r, p) \Rightarrow P(p, q, r)$.

Axiom 5 *(transitivity).* $P(p, q, r) \wedge P(p, q, s) \wedge P(p, q, t) \wedge P(p, r, s) \wedge P(p, s, t) \Rightarrow P(p, r, t)$.

These are fulfilled by all point sets in the Euclidean plane, with $P = \nabla$ defined as point triple orientation. For these sets, the CC Systems are equivalent to the point set order types. However, there exist CC Systems that cannot be constructed as point sets in $\mathbb{R}^2$. These are called *non-realizable* systems, see Section 3.2. CC Systems are equivalent to *abstract order types*, which, for example, can be used in so-called *abstract order type extension* [5]. Every abstract order type can be mapped to an arrangement of pseudo-lines in the projective plane. A stretchable arrangement corresponds to a set of realizable order types [3, pp. 34–35]. Goodman and Pollack showed that all CC Systems of up to 8 elements are realizable as point sets [6]. This fact is useful to show properties of small sets by geometric reasoning.

Theorem 1 *(Goodman, Pollack). Any arrangement of eight pseudo-lines is stretchable.*

The concept of the convex hull of a point set generalizes to all CC Systems. The axiomatic approach can also be extended to cover Delaunay triangulations. In the axiomatic settings, Knuth provides $O(n \log n)$ time algorithms for both problems, where the time bound for the latter holds in the expected case. He points out that the algorithm of Guibas and Stolfi uses the coordinate representation to find a line that partitions the point set into two equally sized subsets (cf. [4, pp. 110–111]). Open Problem 1 in [3, pp. 97–98] therefore asks for an algorithm to find such a partition of a CC System in linear time. The problem is straightforward when given an extreme point of the set (i.e., an element of the convex hull boundary). Proving the existence of a linear-time algorithm for finding a single extreme point is also explicitly part of Open Problem 1.

In this work, we answer both parts of the open problem in the affirmative. In Section 2, we give a simple $O(n)$ time algorithm that, given a point c of a set S of size n, finds a halving edge through c; more specifically, it finds a second point $c' \in S$ s.t. not more than $\lceil \frac{n-2}{2} \rceil$ points are on each side of the supporting line of c and c'. We then describe in Section 3 an algorithm that, given two points p and q, returns the edge of the convex hull that is crossed by the ray from p through q. We first show that the algorithm runs in $O(n)$ time for realizable sets. We then show that the time bound is also correct for non-realizable sets, that is, for all CC Systems. Both algorithms use a prune and search approach.

Our main motivation is to show that the asymptotic running time for solving these problems does not depend on the representation by coordinates. While an arbitrary halving edge can easily be found by picking a point with median, say, x-coordinate, the problem is more sophisticated when the halving line should pass through a predefined point. E.g., the linear time Ham-Sandwich-Cut algorithm of Lo, Matoušek and Steiger [7] can be adapted to find a halving line through a point. The straightforward way of finding an extreme point of a set given by coordinates is selecting the one with, say, lowest x-coordinate. Finding a convex hull edge that is traversed by a given line in linear time is a subroutine of the so-called