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A sufficient condition for the existence of plane spanning trees on geometric graphs ☆

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ABSTRACT

Let P be a set of $n \geqslant 3$ points in general position in the plane and let G be a geometric graph with vertex set P. If the number of empty triangles Δuvw in P for which the subgraph of G induced by $\{u, v, w\}$ is not connected is at most $n - 3$, then G contains a non-self intersecting spanning tree.

1. Introduction

Throughout this article P denotes a set of $n \geqslant 3$ points in general position in the Euclidean plane. A geometric graph with vertex set P is a graph G drawn in such a way that each edge is a straight line segment with both ends in P. A *plane spanning tree* of G is a non-self intersecting subtree of G that contains every vertex of G. Plane spanning trees with or without specific conditions have been studied by various authors.

A well known result of Károlyi et al. [3] asserts that if the edges of a finite complete geometric graph GK_n are coloured by two colours, then there exists a plane spanning tree of GK_n all of whose edges are of the same colour. Keller et al. [4] characterized those plane spanning trees T of GK_n such that the complement graph T^c contains no plane spanning trees.

A plane spanning tree T is a *geometric independency tree* if for each pair $\{u, v\}$ of leaves of T, there is an edge xy of T such that the segments uv and xy cross each other. Kaneko et al. [2] proved that every complete geometric graph with $n \geqslant 5$ vertices contains a geometric independency tree with at least $\frac{n}{6}$ leaves.

Let k be an integer with $2 \leqslant k \leqslant 5$ and G be a geometric graph with $n \geqslant k$ vertices such that all geometric subgraphs of G induced by k vertices have a plane spanning tree. Rivera-Campo [6] proved that G has a plane spanning tree.

Three points u, v and w in P form an *empty triangle* if no point of P lies in the interior of the triangle Δuvw. For any geometric graph G with vertex set P we say that an empty triangle Δuvw of P is *disconnected* in G if the subgraph of G induced by $\{u, v, w\}$ is not connected.

Let $s(G)$ denote the number of disconnected empty triangles of G. Our result is the following:

Theorem 1. *If G is a geometric graph with $n \geqslant 3$ vertices such that $s(G) \leqslant n - 3$, then G has a plane spanning tree.*

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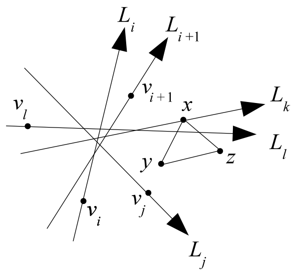

Fig. 1. $x, y, z \in L_i^+ \cap L_{i+1}^+ \cap \cdots \cap L_k^+$, $x = v_k$ and L_l crosses $\{x, y, z\}$.

For each $n \geqslant 3$, let $u_1, u_2, \ldots, u_n$ be the vertices of a regular n-gon and denote by T_n and T_n^c the plane path $u_1, u_2, \ldots, u_n$ and its complement, respectively. The geometric graph T_n^c contains no plane spanning tree and is such that $s(T_n^c) = n - 2$. This shows that the condition in Theorem 1 is tight.

2. Proof of Theorem 1

For every oriented straight line L we denote by L^- the set of points in P which are on or to the left of L and by L^+ the points which are on or to the right of L.

A k-set of P is a subset X of P with k elements that can be obtained by intersecting P with an open half plane. The main tool in the proof of Theorem 1 is the following procedure of Erdős et al. [5,1], used to generate all k-sets of P: Let $L = L_1$ be an oriented line passing through precisely one point v_1 of P with $|L_1^-| = k + 1$. Rotate L clockwise around the axis v_1 by an angle θ until a point v_2 in P is reached. Now rotate L in the same direction but around v_2 until a point v_3 in P is reached, and continue rotating L in a similar fashion obtaining a set of oriented lines $C(L)$ and a sequence of points $v_1, v_2, \ldots, v_s$, not necessarily distinct, where $v_s = v_1$ when the angle of rotation θ reaches 2π.

For $i = 1, 2, \ldots, s - 1$, let $L(v_i, v_{i+1})$ be the line in $C(L)$ that passes through points v_i and v_{i+1} and for $i = 2, 3, \ldots, s - 1$, let L_i be any line in $C(L)$ between $L(v_{i-1}, v_i)$ and $L(v_i, v_{i+1})$.

It is well known that for each line L_j either $L_{j+1}^+ = L_j^+$ and $L_{j+1}^- = (L_j^- \setminus \{v_j\}) \cup \{v_{j+1}\}$, or $L_{j+1}^+ = (L_j^+ \setminus \{v_j\}) \cup \{v_{j+1}\}$ and $L_{j+1}^- = L_j^-$. In both cases $|L_{j+1}^-| = |L_j^-| = k + 1$ and $|L_{j+1}^+| = |L_j^+| = n - k$. It is also easy to see that if $v_{j+1} \in L_j^+$, then $L^-(v_j, v_{j+1}) = L_j^- \cup \{v_{j+1}\}$ and $L^+(v_j, v_{j+1}) = L_j^+$, and if $v_{j+1} \in L_j^-$, then $L^-(v_j, v_{j+1}) = L_j^-$ and $L^+(v_j, v_{j+1}) = (L_j^+ \setminus \{v_j\}) \cup \{v_{j+1}\}$.

The following lemma will used in the proof of Theorem 1.

Lemma 2. *Let $L_i, L_j \in C(L)$ with $i < j$. If x, y and z are points of P lying in $L_i^+ \cap L_j^-$, then there are integers k and l with $i \leqslant k < l < j$ such that $v_k \in \{x, y, z\}$, $x, y, z \in L_k^+ \cap L_j^-$ and such that L_l crosses the triangle Δxyz.*

Proof. Consider the lines $L_i, L_{i+1}, \ldots, L_j$. The result follows from the fact that at each step t, at most one of the points x, y, z switches from L_t^+ to L_{t+1}^-. See Fig. 1. $\square$

Let G be a geometric graph with $n \geqslant 3$ vertices such that $s(G) \leqslant n - 3$ and let P denote the vertex set of G. If $n = 3$ or $n = 4$, it is not difficult to verify by inspection that G has a plane spanning tree. Let us proceed with the proof of Theorem 1 by induction and assume $n \geqslant 5$ and that the result is valid for each geometric subgraph of G with k vertices, where $3 \leqslant k \leqslant n - 1$.

Let v_1 be a point in P and L_1 be an oriented line through v_1 such that $|L_1^-| = \lceil \frac{n+1}{2} \rceil$ and $|L_1^+| = \lfloor \frac{n+1}{2} \rfloor$. Let $C(L)$ be the set of oriented lines obtained from $L = L_1$ as above.

For every $i \geqslant 1$, define G_i^- and G_i^+ as the geometric subgraphs of G induced by L_i^- and L_i^+ respectively, and $G^-(v_i, v_{i+1})$ and $G^+(v_i, v_{i+1})$ as the geometric subgraphs of G induced by $L^-(v_i, v_{i+1})$ and $L^+(v_i, v_{i+1})$, respectively.

We show there is a line in $C(L)$ for which induction applies to the corresponding graphs G^- and G^+, giving plane spanning trees T^- of G^- and T^+ of G^+. As T^- and T^+ lie in opposite sides of L, their union contains a plane spanning tree of G. We analyse several cases.

Case 1. $s(G_1^-) \leqslant |L_1^-| - 3$ and $s(G_1^+) \leqslant |L_1^+| - 3$.

By induction there exist plane spanning trees T_1^- of G_1^- and T_1^+ of G_1^+. Since T_1^- and T_1^+ lie in opposite sides of L_1 and contain exactly one point in common, the graph $T_1^- \cup T_1^+$ is a plane spanning tree of G.

Case 2. $s(G_1^-) \geqslant |L_1^-| - 2$ and $s(G_1^+) \geqslant |L_1^+| - 2$.

Clearly $s(G_1^-) + s(G_1^+) \geqslant (|L_1^-| - 2) + (|L_1^+| - 2) = n - 3 \geqslant s(G) \geqslant s(G_1^-) + s(G_1^+)$. This implies $s(G_1^-) = |L_1^-| - 2$, $s(G_1^+) = |L_1^+| - 2$ and that L_1 does not cross any disconnected empty triangle of G.