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Lower bounds for the number of small convex k-holes ☆

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ABSTRACT

Let S be a set of n points in the plane in general position, that is, no three points of S are on a line. We consider an Erdős-type question on the least number $h_k(n)$ of convex k-holes in S, and give improved lower bounds on $h_k(n)$, for $3 \leqslant k \leqslant 5$. Specifically, we show that $h_3(n) \geqslant n^2 - \frac{32n}{7} + \frac{22}{7}$, $h_4(n) \geqslant \frac{n^2}{2} - \frac{9n}{4} - o(n)$, and $h_5(n) \geqslant \frac{3n}{4} - o(n)$. We further settle several questions on sets of 12 points posed by Dehnhardt in 1987.

0. Introduction

Let S be a set of n points in the plane in general position, that is, no three points of S lie on a common (straight) line. A *k-hole* of S is a simple polygon, P, spanned by k points from S, such that no other point of S is contained in the interior of P. A classical existence question raised by Erdős [10] is: "What is the smallest integer $h(k)$ such that any set of $h(k)$ points in the plane contains at least one convex k-hole?". Esther Klein observed that every set of 5 points contains a convex 4-hole, and Harborth [14] showed that every set of 10 points determines a convex 5-hole. Both bounds are tight w.r.t. the cardinality of S. Only in 2007 and 2008 Nicolás [16] and independently Gerken [13] proved that every sufficiently large point set contains a convex 6-hole. On the other hand, Horton [15] showed that there exist arbitrarily large sets which do not contain any convex 7-hole; see [1] for a brief survey.

A generalization of Erdős' question is: "What is the least number $h_k(n)$ of convex k-holes determined by any set of n points in the plane?" In this paper we concentrate on this question for $3 \leqslant k \leqslant 5$, that is, the number of empty triangles (3-holes), convex 4-holes, and convex 5-holes. We denote by $h_k(S)$ the number of convex k-holes determined by S, and by $h_k(n) = \min_{|S|=n} h_k(S)$ the number of convex k-holes any set of n points in general position must have. Throughout this

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Table 1
The updated bounds on $h_5(n)$ for small values of n.

n	10	11	12	13	14	15	16	17	18
$h_5(n)$	1	2	3	3..4	3..6	3..9	$\geqslant 3$	$\geqslant 4$	$\geqslant 5$
n	19..23	24	25	26..30	31	32	33..37	38	39
$h_5(n)$	$\geqslant 6$	$\geqslant 7$	$\geqslant 8$	$\geqslant 9$	$\geqslant 10$	$\geqslant 11$	$\geqslant 12$	$\geqslant 13$	$\geqslant 14$
n	40..44	45	46	47..50	51	52	53	54..56	57
$h_5(n)$	$\geqslant 15$	$\geqslant 16$	$\geqslant 17$	$\geqslant 18$	$\geqslant 19$	$\geqslant 19$	$\geqslant 20$	$\geqslant 21$	$\geqslant 22$

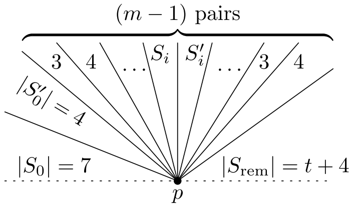

Fig. 1. Partition of $S \setminus \{p\}$ clockwise around an extreme point p: starting with the pair S_0, S'_0; continuing with $(m-1)$ pairs of sets S_i, S'_i, for $1 \leqslant i \leqslant (m-1)$, with $|S_i| = 3$ and $|S'_i| = 4$; and ending with the remainder set S_{rem}.

paper let $\operatorname{ld} x = \frac{\log x}{\log 2}$ be the binary logarithm (logarithmus dualis). Furthermore, we denote with $\operatorname{CH}(S)$ the convex hull of S and with $\partial \operatorname{CH}(S)$ the boundary of $\operatorname{CH}(S)$.

We start in Section 1 by providing improved bounds on the number of convex 5-holes. In particular, we increase the previously best known bound $h_5(n) \geqslant \frac{n}{2} - O(1)$ by Valtr [18] to $h_5(n) \geqslant \frac{3n}{4} - n^{0.87447} + 1.875$. In Section 2 we combine these results with a technique recently introduced by García [11,12], and improve the previously best bounds on the number of empty triangles and convex 4-holes, $h_3(n) \geqslant n^2 - \frac{37n}{8} + \frac{23}{8}$ and $h_4(n) \geqslant \frac{n^2}{2} - \frac{11n}{4} - \frac{9}{4}$ (both in [12]), to $h_3(n) \geqslant n^2 - \frac{32n}{7} + \frac{22}{7}$ and $h_4(n) \geqslant \frac{n^2}{2} - \frac{9n}{4} - 1.2641n^{0.926} + \frac{199}{24}$, respectively. In Section 3 we use these results to answer several questions on sets of 12 points posed by Dehnhardt [8] in 1987.

1. Convex 5-holes

The currently best upper bound on the number of convex 5-holes, $h_5(n) \leqslant 1.0207n^2 + o(n^2)$, is by Bárány and Valtr [7], and it is widely conjectured that $h_5(n)$ grows quadratically. Still, to this date not even a super-linear lower bound is known.

As early as in 1987 Dehnhardt presented a lower bound of $h_5(n) \geqslant 3\lfloor \frac{n}{12} \rfloor$ in his thesis [8]. Unfortunately, this result, published in German only, remained unknown to the scientific community until recently. Thus, the best known lower bound was $h_5(n) \geqslant \lfloor \frac{n}{10} \rfloor$, published by Bárány and Füredi [5] in 1987, later (in 2001) refined to $h_5(n) \geqslant \lfloor \frac{n-4}{6} \rfloor$ by Bárány and Károlyi [6]. Both bounds are derived from the result of Harborth [14]. In the presentation of [11] the lower bound was improved to $h_5(n) \geqslant \frac{2}{9}n - \frac{25}{9}$. A slightly better bound $h_5(n) \geqslant 3\lfloor \frac{n-4}{8} \rfloor$ was presented in [3], which was then sharpened to $h_5(n) \geqslant \lceil \frac{3}{7}(n-11) \rceil$ in [4]. The latest and so far best bound of $h_5(n) \geqslant \frac{n}{2} - O(1)$ is due to Valtr [18]. In this section we further improve this bound to $h_5(n) \geqslant \frac{3}{4}n - o(n)$.

We start by fine-tuning the proof from [4], showing $h_5(n) \geqslant \lceil \frac{3}{7}(n-11) \rceil$, by utilizing the results $h_5(10) = 1$ [14], $h_5(11) = 2$ [8], and $h_5(12) \geqslant 3$ [8]. Although this does not lead to an improved lower bound of $h_5(n)$ for large n, it provides better lower bounds for small values of n, $17 \leqslant n \leqslant 56$; see Table 1.

Lemma 1. *Every set S of n points in the plane in general position with $n = 7 \cdot m + 9 + t$ (for any natural number $m \geqslant 0$ and $t \in \{1, 2, 3\}$) contains at least $h_5(n) \geqslant 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.*

Proof. Because of $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) \geqslant 3$ this is true for $m = 0$. Obviously $h_5(n) \geqslant h_5(n-1)$. Hence, $h_5(n) \geqslant 3$ for any $n \geqslant 12$.

If there exists a point $p \in (\partial \operatorname{CH}(S) \cap S)$ that is a point of a convex 5-hole, then $h_5(S) \geqslant 1 + h_5(S \setminus \{p\}) \geqslant 1 + h_5(n-1)$. In this case, the lemma is true by induction, as for $t = 1$ and $m > 0$, $h_5(n-1) = h_5(7 \cdot m + 9) \geqslant h_5(7 \cdot (m-1) + 9 + 3)$. (The case $t \in \{2, 3\}$ follows trivially, as $h_5(n-1) = h_5(7 \cdot m + 9 + (t-1))$ and $(t-1) \in \{1, 2\}$.)

Otherwise, no point $p \in (\partial \operatorname{CH}(S) \cap S)$ is a point of a convex 5-hole. For $m > 0$ choose one such point p (e.g. the bottom-most one) and successively partition $S \setminus \{p\}$ (in clockwise order around p) into the following (disjoint) subsets: S_0 containing the first 7 points; S'_0 containing the next 4 points; $(m-1)$ pairs of subsets S_i, S'_i: S_i containing 3 points and S'_i containing 4 points $(1 \leqslant i \leqslant (m-1))$; and the subset S_{rem} containing the remaining $(t+4)$ points. See Fig. 1 for a sketch.

The union $S_0 \cup S'_0 \cup \{p\}$ (of disjoint subsets) has cardinality 12 and thus contains at least 3 convex 5-holes [8]. The same is true for each union $S'_{i-1} \cup S_i \cup S'_i \cup \{p\}$ $(1 \leqslant i \leqslant (m-1))$. Finally, the union $S'_{m-1} \cup S_{\text{rem}} \cup \{p\}$ has cardinality $(9+t)$ and therefore contains at least t convex 5-holes [8,14]. Note that we count every convex 5-hole at most once, as the considered