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Local transformations of hexahedral meshes of lower valence

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The modification of conforming hexahedral meshes is difficult to perform since their structure does not allow easy local refinement or un-refinement such that the modification does not go through the boundary. In this paper we prove that the set of hex flipping transformations of Bern et al. (2002) [1] is the only possible local modification on a geometrical hex mesh of valence less than five i.e., with less than five edges per vertex. We propose a new basis of transformations that can generate sequences of local modifications on hex meshes of valence less than six. For quadrilateral meshes, we show the equivalence between modifying locally the number of quads on a mesh and the number of its internal vertices.

1. Introduction

Many industrial codes performing numerical simulations use hexahedral and quadrilateral meshes. For example in mechanics, hexahedral and quadrilateral meshes are preferred to tetrahedral or triangle meshes by some scientists. An important field of computational fluid dynamics (CFD) is the accuracy of the result obtained from a numerical scheme. One of the means used to optimize the accuracy or cost of a numerical scheme in CFD is mesh adaptation. For a given numerical scheme the aim of mesh adaptation is to change locally the size of the mesh with respect to the behavior of the physics being solved. But one difficulty with the modification of hexahedral meshes is to transform it locally without touching a large region of the mesh, due to its structure as a set of layers.

In [1], Bern et al. proposed flipping transformations for hexes which are local modifications of hex meshes i.e., the modifications does not change the boundary quads of the local domain where they are applied. No other local modifications of conforming and geometrical hex meshes of valence less than five exist to our knowledge. The flips for hex meshes of Bern et al. were obtained as possible exchanges of upper and lower facets of a four-dimensional cube.

We propose a characterization of the local modifications on hex meshes of valence less than five. We prove combinatorially that the flips for hex meshes are the only possible local modifications on a geometrical hex mesh of valence less than five, using relationships between vertices, faces, edges and cells of the mesh. We also propose an extension of the flips for hex meshes to modifications involving meshes of valence less than six.

Local modifications on quad meshes seem to be easier to perform than on hex meshes. We propose a characterization of these operations as equivalent to the modification of the number of internal vertices of the quad mesh.

The paper is organized as follows. The second section is devoted to the definitions and notations. In the third section, we prove that the modification of the number of quads on a quad mesh with its boundary unchanged is equivalent to the modification of the number of its internal vertices. We also propose a characterization of local modifications of hex meshes of valence less than five. In the fourth section, we show that the hex flipping transformations of Bern et al. are the unique

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set of local transformations of hex meshes of valence less than five. In the fifth section, we propose an extension of the flips for hex meshes to a new set of local transformations of geometrical hex meshes with less than 6 edges per vertex, and which generates sequences of transformations.

2. Definitions and notations

A *quad surface mesh* is a polyhedron whose faces are all quads. In what follows, *boundary* means part of the surface mesh, and *internal* is the complement. A *hex* (hexahedron) *mesh* is a decomposition of the interior of a polyhedron into hexahedra, where two hexes intersect at a *facet* or not at all, and the boundary of the hex mesh is a quadrilateral mesh of the polyhedral surface. We denote by *facet* of a hex mesh: a vertex, an edge, a quad, or a hex.

Quad surface meshes and hex meshes are both examples of *cubical complexes* [2,6,7], the analogue of simplicial complexes in which each k-dimensional face is a k-dimensional cuboid, i.e., a combinatorial cube in k dimensions. The *valence* of a vertex is the number of edges connected to that vertex. An $M_{e/v}$-*hex mesh* is a conforming hex mesh without holes, of which every vertex is connected to M edges at the most. We denote by *cell* a quadrilateral on a quad mesh or a hexahedron on a hex mesh. We also denote by *face* an edge on a quad mesh or a quadrilateral on a hex mesh. The *body* of a hex mesh is all facets which have no vertex on the boundary.

In the following, if not mentioned, we consider only *non-singular* simply connected meshes (without holes): simply connected meshes of which each cell has at least one common face with the rest of the mesh.

On a mesh, we denote by P the number of hexes; F the number of faces (quads); E the number of edges; V the number of vertices; V_j^k the number of vertices with k boundary edges and j internal edges. For facet type f, $\bar{f}$ denotes the number of boundary f, and $\mathring{f}$ denotes the number of internal f. Both quad meshes and hex meshes are considered in the following sections.

For ball topologies, we use Euler's and Cauchy's formulas [5,3]:

$$V - E + F - P = 1, \tag{2.1}$$

for a complex of polyhedra, and

$$V - E + F = 2, \tag{2.2}$$

for a single polyhedron.

3. Local modifications of quadrilateral and hexahedral meshes

3.1. On quadrilateral meshes

Changing the number of quads of a quadrilateral mesh while keeping unchanged its boundary, is equivalent to changing only the number of its internal vertices.

Theorem 3.1. *The variation of the number of quads equals the variation of the number of internal vertices between two quad meshes* ($mesh_a$ *and* $mesh_b$) *of the same domain, if and only if the two meshes have the same number of boundary vertices*:

$$\mathring{V}_a - \mathring{V}_b = F_a - F_b \quad \Longleftrightarrow \quad \bar{V}_a = \bar{V}_b.$$

Proof. Consider a quad mesh. Thanks to the Euler formula [5],

$$V - E + F = 1. \tag{3.1.1}$$

On a quad mesh each internal edge is shared by exactly two quads and the number of edges and vertices are the same on the boundary so, we have:

$$\left.\begin{array}{l} 4F = 2\mathring{E} + \bar{E} \\ \bar{E} = \bar{V} \end{array}\right\} \quad \Longrightarrow \quad \mathring{E} = 2F - \frac{\bar{V}}{2}.$$

(3.1.1) gives: $\mathring{V} + \bar{V} - (\mathring{E} + \bar{E}) + F = 1.$

$$\Longrightarrow \quad \mathring{V} - F = 1 - \frac{\bar{V}}{2}. \tag{3.1.2}$$

By (3.1.2), $\mathring{V}_a - \mathring{V}_b = F_a - F_b$

iff $\mathring{V}_a - F_a = \mathring{V}_b - F_b$

iff $1 - \frac{\bar{V}_a}{2} = 1 - \frac{\bar{V}_b}{2}$

iff $\bar{V}_a = \bar{V}_b.$ □