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Approximating minimum bending energy path in a simple corridor ☆

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ABSTRACT

In this paper, we consider the problem of computing a minimum bending energy path (or MinBEP) in a simple corridor. Given a simple 2D corridor C bounded by straight line segments and arcs of radius $2r$, the MinBEP problem is to compute a path P inside C and crossing two pre-specified points s and t located at each end of C so that the bending energy of P is minimized. For this problem, we first show how to lower bound the bending energy of an optimal curve with bounded curvature, and then use this lower bound to design a $(1+\epsilon)$-approximation algorithm for this restricted version of the MinBEP problem. Our algorithm is based on a number of interesting geometric observations and approximation techniques on smooth curves, and can be easily implemented for practical purpose. It is the first algorithm with a guaranteed performance ratio for the MinBEP problem.

1. Introduction

In this paper, we consider the following minimum bending energy path (MinBEP) problem: Given a simple corridor C bounded by the straight line segments and arcs of radius $2r$ (a precise definition of simple corridor is given later) and two points s and t at each end of C, compute a minimum energy path P crossing s and t and traversing the corridor C. The energy of a smooth path P is measured by its bending energy E_b.

The bending energy was suggested by Bernoulli in 1738 and studied intensively for the case of planar curves by Euler [10]. Given a smooth parameterized planar curve s, its bending energy is defined as the integral of the squared curvature over the length of the curve, i.e., $E_b = \int_s \kappa^2 \, ds$, where κ is the curvature of the curve at point s. Let $s(t) = (x(t), y(t))$ be the parametrization of an arbitrary point on a smooth curve. Then the curvature [26] of the curve at $s(t)$ is given by $\kappa(t) = \frac{\dot{x}(t)\ddot{y}(t)-\dot{y}(t)\ddot{x}(t)}{(\dot{x}(t)^2+\dot{y}(t)^2)^{3/2}}$, where $\dot{x}(t)$ is the first derivative of $x(t)$ and $\ddot{y}(t)$ is the secondary derivative of $y(t)$. Intuitively, the curvature of a smooth curve at point $s(t)$ is defined as the inverse of its minimum curvature radius (i.e., the radius of the osculating circle at $s(t)$).

The MinBEP problem considered in this paper is motivated by applications in interventional procedures for cardiovascular surgery [17,24,25,27]. In such applications, an elastic wire (called *guidewire*) is often used to guide a catheter/stent through the blood vessels to the narrowed (stenosed) spot to open it up. To optimize the interventional procedure (e.g., selecting the best fitting stent and catheter), it is often desirable to compute the path of the guidewire in advance. To solve this problem, one way is to view the blood vessel as a narrow corridor and determine the path of a stabilized guidewire by computing

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the minimum bending energy path inside the corridor. Thus the guidewire problem can be formulated as a MinBEP problem in 3D. In this paper we consider the 2D case. The framework of our approach is potentially applicable to the 3D case.

Due to the wide use of elastic curves in many fields, the minimum energy path problem has been extensively studied in the past. Most of the studies have focused on computing closed or knotted elastic curves with prescribed boundary conditions and fixed length [12,16,20,21]. Almost all of them are based on numerical techniques (e.g., partial differential equations). For unknotted curves, several results are known for solving the guidewire problem [17,24,25,27]. Among them, the works in [17,27] are based on finite element methods (FEM) and thus converge rather slowly. The work in [24,25] introduces a heuristic algorithm to approximate the path of a guidewire. It demonstrates much improved time efficiency (e.g., seconds vs. hours or even days needed by the FEM approaches). However, due to its heuristic nature, it does not have a provable quality guarantee and the resulting path is in general not smooth. Bending energy has also been used in computational biology for understanding the cellular processes [15]. In addition, in shape completion which is an intriguing problem in geometry processing with applications in CAD and graphics, a 3D Euler spiral — the curve having both its curvature and torsion evolve linearly along the curve is proposed and proved to be existent and unique up to a rigid transformation [14].

A problem closely related to the MinBEP problem is the curvature-bounded shortest path (also called minimum cost path) problem. It has been frequently used for motion planning in robotics and artificial intelligence and extensively studied [1–3,5,6,9,11,19]. In [22], it has been shown that in arbitrary polygons, the shortest path problem with constrained curvature is NP-Hard. An exponential time and space algorithm for generating feasible solutions is given in [11]. More efficient approximation algorithms were also obtained in [4,18,28].

In this paper, we consider the problem of approximating the 2D MinBEP problem. To fix the problems associated with existing solutions (e.g., [17,24,25,27,29]), we expect that the computed path (1) has a bending energy close to optimal, (2) is smooth at every point, and (3) can be computed efficiently. To achieve these goals, our strategy is to first solve a constrained version of the problem. In this constrained version, we require that the to-be-computed path P have a bounded curvature $\kappa_{\max}$ depending on the shape of the corridor. Once we obtain solutions to the curvature bounded MinBEP problem, we can then perform a binary search on the maximum curvature to find the best solution to the MinBEP problem. Thus our focus is on the curvature bounded minimum bending energy path problem.

We note that even this constrained MinBEP problem is extremely challenging to solve optimally for several reasons. Firstly, the curvature of an optimal curve could change continuously between 0 and $\kappa_{\max}$ along the corridor, which seems to suggest that the problem may not even belong to the class of *NP*. Secondly, the optimal path could appear at any place inside the continuous free space of the corridor, making it rather difficult to predict its shape and position. Thirdly, due to the quadratic nature of the objective function, the solution to any subpath depends on the entire unknown optimal path and cannot be determined locally.

Thus our goal for the MinBEP problem is to obtain a good approximation solution. To deal with the aforementioned difficulties, we have to be able to estimate the bending energy of the optimal path, predict how the energy is distributed along the corridor, and ensure the quality of the whole computed path. For these purposes, we first use a set of recursive shortest paths and local features of the corridor to establish a lower bound on the energy of an optimal solution as well as the possible energy distribution. With this lower bound, we then place various types of grid points to discretize the continuous space and guess the positions of the optimal path. With the help of a number of interesting observations on the minimum bending energy path, sophisticated approximation techniques, and involving analysis on the quality of the path, we show that it is possible to obtain a $(1+\epsilon)$-approximation from the discretized space. Our algorithm runs in nearly quadratic time and can be easily implemented for practical purpose. To our best knowledge, this is the first quality guaranteed solution to the MinBEP problem.

2. Preliminaries

Let D be a disk of radius r. A pipe is a rectangle with a fixed width of $2r$ and a non-fixed length e. An elbow is a sector with a fixed radius of $2r$ and a non-fixed center angle of $\pi - \theta$ for some $0 \leqslant \theta \leqslant \pi$. A corridor is a set of alternatively appearing pipes and elbows concatenated in a way that the radius of each elbow completely overlaps with the side of a neighboring pipe with length $2r$. A corridor C is simple if every pipe or elbow does not intersect the interior of any other pipe or elbow, and non-simple otherwise (see Fig. 1). Clearly, there is no hole in a such defined simple corridor. Each elbow together with its two adjacent pipes forms an elbow region *ER* (see Fig. 2).

Let $\eta = \{v_1, v_2, \ldots, v_n\}$ be the centerline of C. It is easy to see that η is a simple 2D curve, where the edge between every pair of consecutive vertices v_i and v_{i+1} is either a straight line segment or an arc of radius r which alternatively appears along η. Thus C can also be viewed as the Minkowski sum $\eta \oplus D$ of η and D. In geometry, the Minkowski sum (also known as dilation) of two sets of position vectors A and B in Euclidean space is formed by adding each vector in A to each vector in B, i.e., the set $A + B = \{a + b \mid a \in A,\ b \in B\}$. The corridor C is bounded by straight line segments and arcs. Let C_u and C_l be the upper and lower boundary of C, separated from the two ends of C. (Assume that the segment $\overline{v_1 v_n}$ is horizontally oriented.) Let $\{u_1, u_2, \ldots, u_n\}$ and $\{w_1, w_2, \ldots, w_m\}$ be the vertices of C_u and C_l respectively. Thus the edges between consecutive vertices in C_u and C_l are either segments or arcs. It is also easy to see that for each vertex in the boundary of C, if it is adjacent to two segments, then it is a reflex vertex (i.e., the inner angle is larger than π).