Computational Geometry 46 (2013) 879–893

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

On the number of crossing-free partitions

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ARTICLE INFO

Article history:
Received 15 June 2009
Accepted 7 July 2011
Available online 18 July 2011
Communicated by Stefan Langerman

Keywords:
Crossing-free geometric graphs
Crossing-free partitions
Counting

ABSTRACT

A partition of a point set in the plane is called crossing-free, if the convex hulls of the individual parts do not intersect. We prove that convex position of a planar set of n points in general position minimizes the number of crossing-free partitions into 1, 2, 3, and $n-3$, $n-2$, $n-1$, n partition classes. Moreover, we show that for all $n \geqslant 5$ convex position of the underlying point set does not maximize the total number of crossing-free partitions.

It is known that in convex position the number of crossing-free partitions into k classes equals the number of partitions into $n-k+1$ parts. This does not hold in general, and we mention a construction for point sets with significantly more partitions into few classes than into many.

1. Introduction

Let P be a set of n points in the plane. We assume that P is in *general position*, i.e., no three points are collinear. A partition of P is called *crossing-free* if the convex hulls of the individual parts do not intersect. Note that one may uniquely identify such a crossing-free partition with a planar straight-line embedded graph on the vertex set P consisting of the edges forming the boundary of the convex hulls of the partition classes. The converse statement, however, is not true in general, as a planar graph might contain nested convex hull boundaries, and hence is not corresponding to a crossing-free partition. We denote by $\mathrm{cfp}(P)$ the number of crossing-free partitions of P, and write $\mathrm{cfp}_k(P)$ for the number of crossing-free partitions of P into k classes, where $1 \leqslant k \leqslant n$. Moreover, Γ_n denotes a set of n points in convex position, i.e., Γ_n is the vertex set of a convex n-gon.

Crossing-free partitions of Γ_n were first treated by Becker [3]. In his note on *planar rhyme schemes* he mentioned yet another appearance of the well-known Catalan numbers, namely, for any $n \in \mathbb{N}$ points in convex position

$$\mathrm{cfp}(\Gamma_n) = C_n = \frac{1}{n+1}\binom{2n}{n} = \Theta\left(\frac{4^n}{n^{3/2}}\right),$$

where C_n denotes the n-th Catalan number. Furthermore, Kreweras [10] calculated

$$\mathrm{cfp}_k(\Gamma_n) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \frac{(n-1)!n!}{(k-1)!k!(n-k)!(n-k+1)!}, \tag{1}$$

which are also famous integer sequences known as the Narayana numbers. A short proof of both these identities employing the framework of generating functions is given in Flajolet and Noy [6].

We will show that Γ_n minimizes $\mathrm{cfp}_k(P)$ for certain values of k if P is in general position, and in fact we conjecture this statement to be true for all k. Note that the term for $\mathrm{cfp}_k(\Gamma_n)$ is symmetric in the sense that $\mathrm{cfp}_k(\Gamma_n) = \mathrm{cfp}_{n-k+1}(\Gamma_n)$, for

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[1] Research supported by the Swiss National Science Foundation, SNF project 200021-116741.

0925-7721/$ – see front matter
doi:10.1016/j.comgeo.2011.07.001

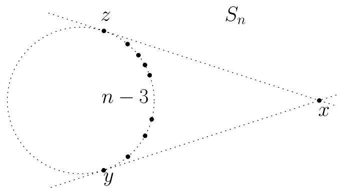

Fig. 1. Construction of the single chain.

all $1 \leqslant k \leqslant n$. We will see that this is not necessarily the case if the points are not in convex position as, for constant $k \geqslant 3$, we mention a construction for sets P_k of n points such that

$$\frac{\mathrm{cfp}_k(P_k)}{\mathrm{cfp}_{n-k+1}(P_k)} = \Omega\big(n^2\big).$$

García et al. [7] proved that Γ_n minimizes the number of perfect matchings and spanning trees among point sets in general position. Note that Γ_n has $C_{n/2}$ many perfect matchings [12]. Aichholzer et al. [2] and [1] extended these results by showing that similar statements also hold for several other graph classes like spanning paths, (pointed) pseudo-triangulations, forests, connected graphs, or all plane graphs. However, it is well known that triangulations are a prominent counter-example to this pattern, a result following from Hurtado and Noy [9]. It is open whether Γ_n minimizes the total number of crossing-free partitions but we conjecture an affirmative answer. Sharir and Welzl [13] show that $\mathrm{cfp}(P) = O(12.24^n)$ and the so-called double-chain, introduced in [7], allows for $\Omega(5.23^n)$ crossing-free partitions. While for n large enough this is certainly more than $\mathrm{cfp}(\Gamma_n)$ it is not true for small n.

Proposition 1. *For every $n \geqslant 5$, there is a set S_n of n points in general position such that*

$$\mathrm{cfp}(S_n) > \mathrm{cfp}(\Gamma_n) = C_n.$$

Proof. Let S_n denote the single chain formed by n points which is given by the following construction. For a given circle and a point x outside let y and z be the two points where the tangents through x touch the circle. Then place $n-3$ points on the circle between y and z, such that the points are contained in the triangle defined by x, y and z. See Fig. 1.

We exhaustively count the number of crossing-free partitions of S_n by separately considering the cases where x belongs to partition classes of size k, for $1 \leqslant k \leqslant n$. By construction, whenever x is in a class of size $k \geqslant 2$ the other $k-1$ points of this class are consecutive points of the convex set $S_n \setminus \{x\}$. Hence, there are $n-(k-1)$ choices for such a partition class. Furthermore, observe that the remaining $n-k$ points are in convex position, and that their individual convex hulls do not intersect the hull of the class containing x. Hence, every crossing-free partition of these $n-k$ points is also crossing-free when additionally considering the partition class containing x. Thus, we find

$$\mathrm{cfp}(S_n) = C_{n-1} + \sum_{k=2}^{n}\big(n-(k-1)\big)C_{n-k} =: s_n.$$

Note that $C_{n+1} = 2\frac{2n+1}{n+2} \cdot C_n < 4C_n$ holds for all $n \in \mathbb{N}$. Then the claim $s_n > C_n$ is easily established for every $n > 13$, since truncating the sum yields $s_n > C_{n-1} + (n-1)C_{n-2} > C_{n-1} + 12C_{n-2} > C_{n-1} + 3C_{n-1} > C_n$. For the remaining values of $5 \leqslant n \leqslant 12$ we simply calculate the exact values of s_n and C_n and verify the statement. □

Let P be a finite set of points in general position and $Q \subseteq P$ a non-empty subset. Then by $E(Q) := Q \cap \partial\,\mathrm{conv}(Q)$ we refer to the extreme points of Q, and the points from P contained inside the convex hull of Q are denoted by $I^P(Q) := P \cap \mathrm{conv}(Q)^\circ$. Here, ∂S denotes the boundary and S° the interior of a set S. For $k \in \mathbb{N}$, we employ the common notion of $\binom{P}{k}$ for the k-element subsets of P. Finally, given a predicate A we write $\mathbb{1}_{[A]}$ for the indicator function, i.e., $\mathbb{1}_{[A]} = 1$ if A holds and $\mathbb{1}_{[A]} = 0$ otherwise.

2. Partitioning into many classes

As a warm-up observe that the number of crossing-free partitions of n points into n and $n-1$ classes does not depend on the relative position of the points, as long as they are in general position. A partition into $n-1$ classes corresponds to a planar graph with exactly one edge.

Proposition 2. *For a set P of n points in the plane in general position* $\mathrm{cfp}_n(P) = 1$ *and* $\mathrm{cfp}_{n-1}(P) = \binom{n}{2}$.