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Witness Gabriel graphs [☆]

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ABSTRACT

We consider a generalization of the Gabriel graph, the *witness Gabriel graph*. Given a set of vertices P and a set of witness points W in the plane, there is an edge ab between two points of P in the witness Gabriel graph $\mathrm{GG}^-(P, W)$ if and only if the closed disk with diameter ab does not contain any witness point (besides possibly a and/or b). We study several properties of the witness Gabriel graph, both as a proximity graph and as a new tool in graph drawing.

1. Introduction

Originally defined to capture some concept of neighborliness, *proximity graphs* [9,12,16] can be intuitively defined as follows: Given a set P of points in the plane, the vertices of the graph, there is an edge between a pair of vertices $p, q \in P$ if some specified region in which they interact contains no point from P, besides possibly p and q.

Proximity graphs have proved to be a very useful tool in shape analysis and in data mining [9,15]. In graph drawing, a problem that has been attracting substantial research is to explore which classes of graphs admit a proximity drawing, for some notion of proximity, and when it is possible to efficiently decide, for a given graph, whether such a drawing exists [4,12]. For all these reasons, several variations and extensions have been considered, from higher-order proximity graphs to the so-called weak proximity drawings [3,9].

In the case of the *Gabriel graph*, $\mathrm{GG}(P)$, the region of influence of a pair of vertices a, b is the closed disk with diameter ab, D_{ab}. An edge ab is in the Gabriel graph of a point set P if and only if $P \cap D_{ab} = \{a, b\}$ (see Fig. 1 (left)). Gabriel graphs were introduced by Gabriel and Sokal [8] in the context of geographic variation analysis.

We consider in this work a generalization of the Gabriel graph, the *witness Gabriel graph*, $\mathrm{GG}^-(P, W)$. It is defined by two sets of points P and W; P is the set of vertices of the graph and W is the set of *witnesses*. There is an edge ab in $\mathrm{GG}^-(P, W)$ if, and only if, there is no point of W in $D_{ab} \setminus \{a, b\}$ (see Fig. 1 (right)).

Notice that witness Gabriel graphs are a proper generalization of Gabriel graphs, because when the set W of witnesses coincides with the set P of vertices, we clearly obtain $\mathrm{GG}^-(P, P) = \mathrm{GG}(P)$. This was the main underlying idea of the generic concept of *witness graphs*, which were introduced as a general framework in [1] to provide a generalization of

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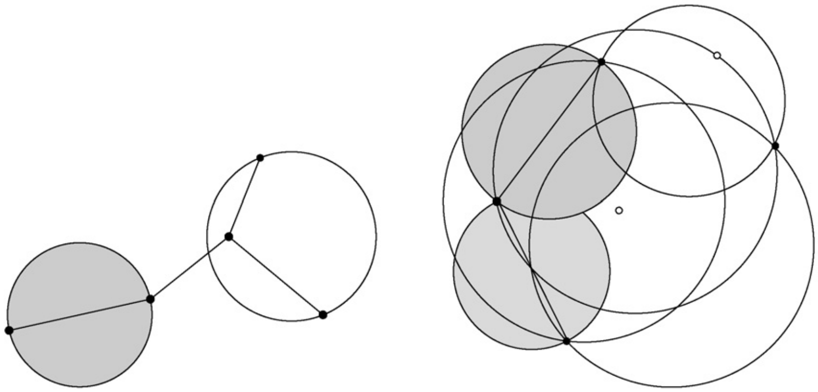

Fig. 1. Left: Gabriel graph. The vertices defining the shaded disk are adjacent because their disk doesn't contain any other vertex, in contrast to the vertices defining the unshaded disk. Right: Witness Gabriel graph. Black points are the vertices of the graph, white points are the witnesses. Each pair of vertices defining a shaded disk are adjacent and the pairs defining the remaining (unshaded) disks are not.

proximity graphs, allowing witnesses to play a negative role as in this paper or a positive one as well. Several examples were described in [1,2], including in particular witness versions of Delaunay graphs and rectangular influence graph. A systematic study is developed in [6]. As already mentioned in this introduction, generalizing basic proximity graphs has attracted several research efforts. This is also our main motivation. On the other hand, a witness graph $W(P, Q)$ is an instrument for describing the position of P with respect to Q. We believe that once these graphs are well understood, by considering simultaneously $W(P, Q)$ and $W(Q, P)$ we would have useful tools for the description of the interaction/discrimination between the two sets; this is a main topic of our ongoing research.

In this paper we prove several fundamental properties of witness Gabriel graphs, describe algorithms for their computation, and present results on the realizability of some combinatorial graphs.

We assume throughout the paper that the points in $P \cup W$ are in general position, that is, that there are no three points of $P \cup W$ on a line and no four on a circle.

2. Some properties of witness Gabriel graphs

It is known that $\mathrm{MST}(P) \subseteq \mathrm{GG}(P) \subseteq \mathrm{DT}(P)$ [10], where $\mathrm{MST}(P)$ is the minimum spanning tree and $\mathrm{DT}(P)$ is the Delaunay triangulation. As a consequence, $|\mathrm{MST}(P)| \leqslant |\mathrm{GG}(P)| \leqslant |\mathrm{DT}(P)|$, where we have used $|\cdot|$ to denote the number of edges in a graph. Expressing this in terms of $n = |P|$, we have that $n - 1 \leqslant |\mathrm{GG}(P)| \leqslant 3n - 6$. In [14], a more detailed analysis gives a tighter upper bound of $3n - 8$.

For witness Gabriel graphs $\mathrm{GG}^-(P, W)$, the situation is quite different, as for any fixed set P of n points, by varying the size of W and the location of the witnesses, the number of edges in $\mathrm{GG}^-(P, W)$ can attain any value from 0 to $\binom{n}{2}$. For example, when $W = \varnothing$, we obviously obtain $\mathrm{GG}^-(P, \varnothing) = K_n$.

Theorem 1. *For any set P of n points in the plane, a witness Gabriel graph $\mathrm{GG}^-(P, W)$ can have any number of edges from 0 to $\binom{n}{2}$ edges, by a suitable choice of the set W of witnesses.*

Proof. Consider any given graph $\mathrm{GG}^-(P, W)$ and take the union U of the diametral disks $D_{p_i p_j}$, $p_i, p_j \in P$, that do not contain a point $q \in W$. The boundary of the union consists of circular arcs $C_{p_i p_j}$ of disks $D_{p_i p_j}$, for some $p_i, p_j \in P$. Put a point $q \in W$ in the relative interior of one such arc $C_{p_i p_j} \setminus \{p_i, p_j\}$. Point q lies in the closed disk $D_{p_i p_j} \setminus \{p_i, p_j\}$. By construction, it lies outside all other disks. Therefore adding q to W would eliminate precisely one edge, namely (p_i, p_j).[3] By iterating this procedure to remove the edges one by one from the witness Gabriel graph, one can see that any number of edges can be attained (see Fig. 2). □

The reverse problem is more interesting: As the witness points can be thought as *interferences* that prevent the points in P from being adjacent, one may wonder how many witnesses are required to completely eliminate all edges in $\mathrm{GG}^-(P, W)$. Trivially, if there is a witness inside each disk D_{ab}, for all $a, b \in P$, then $\mathrm{GG}^-(P, W)$ has no edge. This can be achieved, for instance, by putting a witness close to the midpoint of every pair a, b of points from P, which would give $|W| \leqslant \binom{n}{2}$. In the following theorem we present a much better bound for the number of witnesses necessary to eliminate all edges of $\mathrm{GG}^-(P, W)$.

[3] This choice of q is in some sense degenerate, but q can be moved slightly into the interior of $D_{p_i p_j}$ without affecting the argument.