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Approximate proximity drawings ☆

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ABSTRACT

We introduce and study a generalization of the well-known region of influence proximity drawings, called $(\varepsilon_1, \varepsilon_2)$*-proximity drawings*. Intuitively, given a definition of proximity and two real numbers $\varepsilon_1 \geqslant 0$ and $\varepsilon_2 \geqslant 0$, an $(\varepsilon_1, \varepsilon_2)$-proximity drawing of a graph is a planar straight-line drawing Γ such that: (i) for every pair of adjacent vertices u, v, their proximity region "shrunk" by the multiplicative factor $\frac{1}{1+\varepsilon_1}$ does not contain any vertices of Γ; (ii) for every pair of non-adjacent vertices u, v, their proximity region "expanded" by the factor $(1+\varepsilon_2)$ contains some vertices of Γ other than u and v. In particular, the locations of the vertices in such a drawing do not always completely determine which edges must be present/absent, giving us some freedom of choice. We show that this generalization significantly enlarges the family of representable planar graphs for relevant definitions of proximity drawings, including Gabriel drawings, Delaunay drawings, and β-drawings, even for arbitrarily small values of ε_1 and ε_2. We also study the extremal case of $(0, \varepsilon_2)$-proximity drawings, which generalize the well-known weak proximity drawing paradigm.

1. Introduction and overview

Proximity drawings are straight-line drawings of graphs where any two adjacent vertices are deemed to be close according to some proximity measure, while any two non-adjacent vertices are far from one another by the same measure. Different definitions of proximity give rise to different types of proximity drawings. In the *region of influence* based proximity drawings two vertices u and v are adjacent if and only if some regions of the plane, defined by using the coordinates of u and v, are *empty*, i.e. they do not contain any vertices of the drawing other than, possibly, u and v.

For example, the *Gabriel disk* of two points u and v in the plane is the closed disk having u and v as its antipodal points (cf. Fig. 1(a)) and a *Gabriel drawing* is a planar straight-line drawing Γ such that any two vertices in Γ are connected by an edge if and only if their Gabriel disk is empty of other vertices. Note that any such drawing Γ, viewed as a geometric graph in the plane, coincides with the so-called *Gabriel graph* of the points forming the vertex set of Γ.

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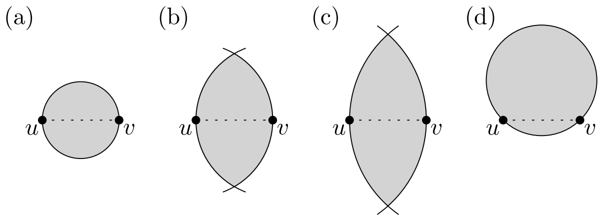

Fig. 1. Two points u and v. Indicated by shading: (a) their Gabriel disk (which coincides with their β-region for $\beta = 1$), (b) their β-region for $\beta = 2$, (c) their β-region for $\beta = 3$, and (d) one of their Delaunay disks.

A generalization of the Gabriel disk is the so-called *β-region of influence* (cf. Fig. 1(b) and (c)): For a given value of β such that $1 \leqslant \beta \leqslant \infty$, the β-region of influence of two vertices u and v having Euclidean distance $d(u, v)$ is the intersection of the two disks of radius $\frac{\beta d(u,v)}{2}$, centered on the line through u and v, one containing u and touching v, the other containing v and touching u (hence the β-region for $\beta = 1$ is the Gabriel disk). Given a value of β, a straight-line drawing Γ is a *β-drawing* if and only if for any edge (u, v) in Γ the β-region of influence of u and v is empty of other vertices, that is, Γ coincides with the so-called *β-skeleton* of the points forming the vertices of Γ.

Delaunay drawings use a definition of proximity that extends the one used for Gabriel drawings. Namely, the *Delaunay disks* of two vertices u and v are the closed disks having $\overline{uv}$ as a chord (the Gabriel disk is therefore a particular Delaunay disk, cf. Fig. 1(d)). In a *Delaunay drawing* Γ an edge (u, v) exists if and only if at least one of the Delaunay disks of u and v is empty of other vertices, that is, Γ coincides with the so-called *Delaunay graph* of its vertex set. Note that the Delaunay graph of a point set P is not a triangulation of P if, for example, more than three points in P lie on a common circle that does not contain other points of P in its interior.

As is not hard to imagine, by changing the definition of region of influence, the combinatorial properties of those graphs that admit a certain type of proximity drawing can change significantly. For example, it is known that not all trees having vertices of degree four admit a Gabriel drawing [5] while they have a β-drawing for $1 < \beta \leqslant 2$ [14]. It should be noted, however, that despite the many papers published on the topic, full combinatorial characterization of proximity drawable graphs remains an elusive goal for most types of regions of influence. The interested reader is referred to [7,13,16] for more references and results on these topics. As a general tendency, using the region of influence based proximity rules recalled above only very restricted families of graphs can be represented. In this paper, we propose a generalization of these rules and show that it considerably extends the families of representable graphs.

1.1. Problem and results

In order to overcome the restrictions on the families of graphs representable as region of influence based proximity drawings, we study graph visualizations that are "good approximations" of these proximity drawings. We want drawings where adjacent vertices are relatively close to each other while non-adjacent vertices are relatively far apart. The idea is to use slightly smaller regions of influence to justify the existence of an edge and slightly larger regions of influence to justify non-adjacent vertices. Note that, since different regions are used to justify the existence and non-existence, respectively, of an edge, it may happen that for a particular pair of vertices in a drawing we actually have the choice to either draw an edge between these vertices or not. In contrast, once the vertex set of a drawing is fixed, the usual proximity rules completely determine which edges must be present/absent in the drawing. This key difference is one of the reasons why larger families of graphs can be represented in the framework presented here. In the following we focus on the representability of various types of planar graphs as this helps us to emphasize the connection with the original proximity rules that necessarily yield plane drawings. Note, however, that our new framework can also represent non-planar graphs.

Now, to describe the modified regions of influence more formally, let D be a disk with center c and radius r, and let ε_1 and ε_2 be two non-negative real numbers. The *ε_1-shrunk disk of D* is the disk centered at c and having radius $\frac{r}{1+\varepsilon_1}$; the *$\varepsilon_2$-expanded disk of D* is the disk centered at c and having radius $(1 + \varepsilon_2)r$. An *$(\varepsilon_1, \varepsilon_2)$-proximity drawing* is a planar straight-line proximity drawing where the region of influence of two adjacent vertices is defined by using ε_1-shrunk disks, while the region of influence of two non-adjacent vertices uses ε_2-expanded disks. Sometimes we will simply refer to such a drawing as an *approximate* proximity drawing.

To illustrate the above definitions, note that all planar graphs (actually, all graphs) with at least one edge or at least three vertices have an $(\varepsilon_1, \varepsilon_2)$-proximity drawing for sufficiently large values of ε_1, ε_2. For example, every planar straight-line drawing Γ of such a graph is an (∞, ∞)-Gabriel drawing since an ∞-shrunk Gabriel disk reduces to a point (and thus the ∞-shrunk disk of every edge in Γ is empty) and an ∞-expanded Gabriel disk is the whole plane (and thus the ∞-expanded disk of any pair of non-adjacent vertices of Γ contains a third vertex, if the graph contains at least three vertices). At the other extreme, a $(0, 0)$-Gabriel drawing is a Gabriel drawing, since a 0-shrunk Gabriel disk is a Gabriel disk and so is a 0-expanded Gabriel disk. Hence, not all planar graphs admit a $(0, 0)$-Gabriel drawing [5].

Based on this observation, our main target is to establish values of ε_1 and of ε_2 that make it possible to compute $(\varepsilon_1, \varepsilon_2)$-proximity drawings for meaningful families of planar graphs and embedded planar graphs. To this end, recall that, for a planar graph G, a *(planar) embedding* of G specifies which face of G is the outer face and, for every vertex v of G,