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Abstract Voronoi diagrams revisited

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ABSTRACT

Abstract Voronoi diagrams [R. Klein, Concrete and Abstract Voronoi Diagrams, Lecture Notes in Computer Science, vol. 400, Springer-Verlag, 1987] were designed as a unifying concept that should include as many concrete types of diagrams as possible. To ensure that abstract Voronoi diagrams, built from given sets of bisecting curves, are finite graphs, it was required that any two bisecting curves intersect only finitely often; this axiom was a cornerstone of the theory. In [A.G. Corbalan, M. Mazon, T. Recio, Geometry of bisectors for strictly convex distance functions, International Journal of Computational Geometry and Applications 6 (1) (1996) 45–58], Corbalan et al. gave an example of a smooth convex distance function whose bisectors have infinitely many intersections, so that it was not covered by the existing AVD theory. In this paper we give a new axiomatic foundation of abstract Voronoi diagrams that works without the finite intersection property.

1. Introduction

Voronoi diagrams belong to the most interesting and useful structures in geometry. Dating back to Descartes [13], and known to mathematicians ever since (see, e.g., Gruber [16]), Voronoi diagrams were the topic of a seminal paper by Shamos and Hoey [30] that helped creating a new field, computational geometry. The general idea is quite natural. There is a space on whom some objects, called sites, exert a certain influence. Each point of the space belongs to the region of that site whose influence is strongest. Most often influence is reciprocal to distance. Meanwhile, CiteSeer lists more than 4800 related articles on Voronoi diagrams. Surveys focussing on their structural and algorithmic aspects were presented by Aurenhammer [6], Aurenhammer and Klein [7], Fortune [15], and, for generalized Voronoi diagrams, by Boissonnat et al. [9]. Beyond their value to computer science, Voronoi diagrams have important applications in many other sciences; prominent examples can be found in Held [17] and in Okabe et al. [28].

For many years, computational geometers have studied Voronoi diagrams in the plane that differed by the types of sites and distance measures used. Typically, algorithms were hand-tailored to fit a particular setting. This situation called for a unifying view. An elegant structural approach was by Edelsbrunner and Seidel [14] who suggested to define general Voronoi diagrams as lower envelopes of suitable "cones". Independently, *Abstract Voronoi Diagrams* (*AVDs*) were introduced by the first author in [19], as a unifying concept for both, structure theory and algorithmic computation.

The basic observation behind AVDs was that Voronoi diagrams are built from systems of bisecting curves that have certain combinatorial properties in common, whereas the nature of the sites and of the distance function are of secondary importance. A challenge was in finding a small set of simple axioms for bisecting curve systems. They should ensure that a Voronoi diagram formed from such a curve system has desirable structural properties (like being a finite plane graph of linear complexity), and that it can be efficiently computed. At the same time, this approach should be as general as possible.

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In order to achieve the goals just mentioned, AVDs were defined in the following way. For any two elements p, q of a set S of indices, also referred to as sites, a curve $J(p,q)$ was given, that splits the plane into two unbounded open domains. One of these domains was labeled by p, the other by q; these labels were part of the definition of $J(p,q)$. The curve itself was added to one of the two domains according to some global order $\prec$ on S. The Voronoi region of p was defined as the intersection of all sets associated with p; detailed definitions will be given in Section 4.

Now three properties were required of the given curves and of order $\prec$. Voronoi regions should be path-connected, and their union should cover the whole plane. Moreover, any two curves $J(p,q)$, $J(r,s)$ should intersect only finitely often. These requirements are met, for example, by the Euclidean Voronoi diagrams of points or line segments, additive weights, power diagrams, and all convex distance functions whose circles are semi-algebraic.

It turned out that these axioms were strong enough to ensure that abstract Voronoi diagrams have many of the properties found in diagrams based on concrete sites and distance functions, and that they can be constructed efficiently.

The finite intersection assumption was instrumental in analyzing the structure of abstract Voronoi diagrams. It was applied twice. First, in proving the topological fact that in a neighborhood of any point v, the bisecting curves passing through v form a star; see the "piece of pie" Lemma 2.3.2 [20]. This fact allowed a local view on which combinatorial definitions could be based. Second, the finite intersection property was explicitly used to guarantee that abstract Voronoi diagrams are finite planar graphs; see Lemma 2.4.2 [20].

Three asymptotically optimal AVD algorithms have been developed, each for a certain subclass of AVDs. A deterministic $O(n \log n)$ divide & conquer algorithm [20], based on work by Shamos and Hoey [30] and by Chew and Drysdale [10], for situations where recursive partitions with cycle-free bisectors are guaranteed; a deterministic linear time algorithm [22] for situations resembling "general convex position", based on the technique by Aggarwal et al. [2], and an $O(n \log n)$ randomized incremental construction algorithm [23] for AVDs whose regions have path-connected interiors, based on work by Clarkson and Shor [11].

McAllister et al. [5], Ahn et al. [3], Karavelas and Yvinec [18], Abellanas et al. [1], Aichholzer et al. [4], and Bae and Chwa [8] presented new types of Voronoi diagrams that were under the umbrella of the AVD concept. The notion of abstract Voronoi diagrams has been generalized to furthest site diagrams by Mehlhorn et al. [27], to dimension 3 by Lê [24], and to a dynamic setting by Malinauskas [25]. A slightly simplified version of abstract Voronoi diagrams has been implemented in LEDA by Seel [31].

But Corbalan et al. [12] gave an example of a convex distance function whose bisectors have an infinite number of intersections; its unit circle is smooth, but not semi-algebraic. The existing AVD concept, with its definitions and proofs relying on the finite intersection property, did not cover this example.

The purpose of this paper is in proving that abstract Voronoi diagrams can be defined and constructed without the finite intersection assumption. In fact, the other two axioms—that Voronoi regions be path-connected and cover the plane—are just strong enough to imply what is needed. Although the proof of this fact requires new techniques quite different from those used in [20,23], we think this effort is well-invested. First, the class of concrete Voronoi diagrams covered by the AVD concept grows; in particular, all convex distance functions are included now. Second, with one axiom less to check, applying AVDs becomes easier. Third, there is scientific value (and aesthetic pleasure) in minimizing axiomatic systems.

Some care is necessary in dealing with general curves that can intersect each other infinitely often. To keep the analysis simple, we require, in this paper, that not only the Voronoi regions, but also their interiors, are path-connected. This was also postulated for the randomized incremental algorithm in [23]. It helped to avoid the complications in [20] that were caused by the fact that Voronoi edges and vertices could form connections between several parts of one Voronoi region. Our requirement also allows us to abandon the order $\prec$, which was used to distribute the bisecting curves among the sites.

The rest of this paper is organized as follows. In Section 2 we state the new set of axioms, and derive some preliminary facts. The main part is Section 3, where we show that AVDs based on the new axioms are finite plane graphs, without resorting to the finite intersection assumption. This is accomplished in the following way. First, we prove that a bisecting curve $J(p,q)$ cannot more than twice alternate between the domains separated by some curve $J(p,r)$, without disconnecting a Voronoi region; see Lemma 6.[1] This allows us to analyze how $J(p,q)$ and $J(p,r)$ can behave in the neighborhood of an intersection point, without having a "piece of pie" lemma available. For sets S of size 3 we show, in Lemma 8, that each point w on the boundary of a Voronoi region is accessible from this region. That is, there exists an arc α with endpoint w such that α without w is fully contained in the Voronoi region.[2]

Using an elegant argument by Thomassen [32], accessibility implies that an abstract Voronoi diagram of three sites contains at most two points that belong to the closure of all Voronoi regions. From this one can directly conclude that AVDs of many sites are finite plane graphs; see Theorem 10. Now a piece of pie lemma can be shown at least for the Voronoi edges meeting at a Voronoi vertex, which is sufficient for our purposes.

In Subsection 3.3 we show that a curve system for index set S fulfills our axioms iff this holds for each subset S' of size three. This fact was observed in [21] for the old AVD model; the proof given in Subsection 3.3 is new and more general.

In Section 4 we address the construction of abstract Voronoi diagram based on the new axioms. With the finite intersection assumption and order $\prec$ removed, the class of curve systems to which randomized incremental construction can be

[1] One should observe that both curves are associated with the same site, p. In the old AVD model [20], this observation was an easy consequence of the fact that AVDs are finite plane graphs.

[2] By the Jordan curve theorem and its inverse, Jordan curves are characterized by accessibility; see Theorem 4.