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On the complexity of umbra and penumbra ☆

J. Demouth [a,*], O. Devillers [b], H. Everett [a], M. Glisse [c], S. Lazard [a], R. Seidel [d]

[a] LORIA, INRIA Nancy - Grand Est, Université Nancy 2, France
[b] INRIA Sophia Antipolis - Méditerranée, France
[c] Gipsa-Lab, CNRS UMR 5216, Grenoble, France
[d] Saarland University, FR Informatik, Saarbrücken, Germany

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ABSTRACT

Computing shadow boundaries is a difficult problem in the case of non-point light sources. A point is in the umbra if it does not see any part of any light source; it is in full light if it sees entirely all the light sources; otherwise, it is in the penumbra. While the common boundary of the penumbra and the full light is well understood, less is known about the boundary of the umbra. In this paper we prove various bounds on the complexity of the umbra and the penumbra cast on a fixed plane by a segment or convex polygonal light source in the presence of convex polygonal or polyhedral obstacles in $\mathbb{R}^3$.

In particular, we show that a single segment light source may cast on a plane, in the presence of two disjoint triangles, four connected components of umbra and that two fat convex and disjoint obstacles of total complexity n can give rise to as many as $\Omega(n)$ connected components of umbra. In a scene consisting of a segment light source and k disjoint convex polyhedra of total complexity n, we prove an $\Omega(nk^2+k^4)$ lower bound on the maximum number of connected components of the umbra and a $O(nk^3)$ upper bound on its complexity; if the obstacles may intersect, we only prove an upper bound of $O(n^2k^2)$.

We also prove that, in the presence of k convex polyhedra of total complexity n, some of which are light sources, the umbra cast on a plane may have in the worst case $\Omega(n^2k^3+nk^5)$ connected components and has complexity $O(n^3k^3)$ (the polyhedra are supposed pairwise disjoint for lower bounds and possibly intersecting for the upper bounds). These are the first bounds on the size of the umbra in terms of both k and n. These results prove that the umbra, which is bounded by arcs of conics, is intrinsically much more intricate than the boundary between full light and penumbra which is bounded by line segments and whose worst-case complexity is, as we show, in $\Omega(nk+k^4)$ and $O(nk\alpha(k)+k^4)$; moreover, if there are only $O(1)$ light sources of total complexity m, then the complexity is in $\Omega(n\alpha(k)+km+k^2)$ and $O(n\alpha(k)+km\alpha(k)+k^2)$.

1. Introduction

Shadows play a central role in human perception [15,20]. Unfortunately, computing realistic shadows efficiently is a difficult problem, particularly in the case of non-point light sources. A wide variety of approaches have been considered for

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* Corresponding author.
E-mail addresses: Julien.Demouth@loria.fr (J. Demouth), Olivier.Devillers@sophia.inria.fr (O. Devillers), Hazel.Everett@loria.fr (H. Everett), marc.glisse@normalesup.org (M. Glisse), Sylvain.Lazard@loria.fr (S. Lazard), rseidel@cs.uni-sb.de (R. Seidel).

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Table 1
Lower bounds on the maximum number of connected components and upper bounds on the complexity of the umbra cast on a plane by one segment light source or polygonal light source(s) in the presence of k convex polyhedra of total complexity $O(n)$.

Scene type	Lower bound	Upper bound
Segment light source:		
2 disjoint triangles	4	$O(1)$
2 fat disjoint convex polyhedra	$\Omega(n)$	$O(n)$
k disjoint convex polyhedra	$\Omega(nk^2 + k^4)$	$O(nk^3)$
k convex polyhedra	$\Omega(nk^2 + k^4)$	$O(n^2k^2)$
Polygonal light source(s):	One light source	$O(k)$ light sources
k convex polyhedra	$\Omega(n^2k^3 + nk^5)$	$O(n^3k^3)$

Table 2
Bounds on the complexity of the union of umbra and penumbra cast on a plane by a set of k convex polyhedra of total complexity n, some of which are light sources.

Light sources	Lower bound	Upper bound
$O(1)$ convex polyhedra of size m	$\Omega(n\alpha(k) + km + k^2)$	$O(n\alpha(k) + km\alpha(k) + k^2)$
$O(k)$ convex polyhedra of total size $O(n)$	$\Omega(nk + k^4)$	$O(nk\alpha(k) + k^4)$

rendering shadows (see, for example, the surveys [8,22]) and many methods make extensive use of graphics hardware (see the survey [13]).

A point is in the *umbra* if it does not see any part of the light source(s); it is in *full light* if it sees entirely all the light source(s); otherwise, it is in the *penumbra*. While the boundary between the penumbra and the full light is reasonably well understood (see Section 3), less is known about the boundary of the umbra. Nevertheless, there is extensive literature concerning the explicit computation of these shadow boundaries; see, for example, [7,9–11,14,16,18,19].

In this paper we prove various bounds, summarized in Tables 1 and 2, on the complexity of the umbra and penumbra cast on a fixed plane by segment or convex polygonal light source(s) in the presence of convex polygonal or polyhedral obstacles in $\mathbb{R}^3$. **Unless specified otherwise, the objects in the scene (light sources and obstacles) are supposed pairwise disjoint for lower bounds and possibly intersecting for upper bounds.** We show, in particular, that a single segment light source may cast, in the presence of two triangles, four connected components of umbra. We prove that the umbra defined by one segment light source and two fat convex obstacles of total complexity n can have as many as $\Omega(n)$ connected components. We also prove an $\Omega(nk^2 + k^4)$ lower bound on the maximum number of connected components of the umbra and a $O(n^2k^2)$ (resp., $O(nk^3)$) upper bound on its complexity in a scene consisting of a segment light source and k possibly intersecting (resp., disjoint) convex polyhedra of total complexity n. Finally, we prove that the umbra cast on a plane by a polygonal light source and k convex obstacles can have $\Omega(n^2k^3 + nk^5)$ connected components and has worst-case complexity $O(n^3k^3)$. These are the first bounds on the size of the umbra in terms of both k and n.

These results are related to work on the *aspect graph*, a structure that encodes all topologically distinct views of a scene, where the definition of the view of a scene depends on the choice of a viewpoint space [5]. Two models of viewpoint space are commonly used: the *orthographic* model, where view points are on the plane at infinity and all lines of sight are parallel to a viewing direction, and the *perspective* model, where view points are in $\mathbb{R}^3$ but not in the objects. In terms of complexity, de Berg et al. [6] proved that a scene consisting of k convex polyhedra of total complexity n has at most $O(n^4k^2)$ distinct orthographic views and at most $O(n^6k^3)$ perspective views. These bounds are tight as shown by Aronov et al. [1]. In our work we limit the viewpoint space to the shadow plane and consider only those views involving the light sources.

Our results are surprising in the sense that they show that the umbra cast by a single segment light source may have many connected components. The fact that the umbra may have four connected components in the case of two triangle obstacles comes as a total surprise. Our lower bounds of $\Omega(nk^2 + k^4)$ and $\Omega(n^2k^3 + nk^5)$ on the maximum number of connected components, for k convex polyhedra of total complexity n, are rather pathological in the sense that most of the obstacles are very long and thin. However, we also present a lower bound example of $\Omega(n)$ connected components in the case of two fat polygons or convex polyhedra of complexity $O(n)$. Concerning the upper bounds of $O(nk^3)$, $O(n^2k^2)$ and $O(n^3k^3)$, even though these bounds are not *a priori* tight, they substantially improve the only previously known bounds for this problem which were the trivial $O(n^4)$ and $O(n^6)$ upper bounds. Finally, it is interesting to point out that even for the simplest case of non-point light sources, obtaining tight bounds on the complexity of the umbra and understanding its structure is a very challenging problem.

These results show that the umbra, which is bounded by arcs of conics, is intrinsically much more intricate than the boundary between full light and penumbra which is bounded by line segments and for which we prove that the worst-case complexity is in $\Omega(nk + k^4)$ and $O(nk\alpha(k) + k^4)$, where $\alpha(k)$ denotes the pseudo-inverse of the Ackermann function; moreover, if there are only $O(1)$ light sources of total complexity m, then the worst-case complexity of the boundary between full light and penumbra is in $\Omega(n\alpha(k) + km + k^2)$ and $O(n\alpha(k) + km\alpha(k) + k^2)$.