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On the choice of test for a unit root when the errors are conditionally heteroskedastic[☆]

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ABSTRACT

It is well known that in the context of the classical regression model with heteroskedastic errors, while ordinary least squares (OLS) is not efficient, the weighted least squares (WLS) and quasi-maximum likelihood (QML) estimators that utilize the information contained in the heteroskedasticity are. In the context of unit root testing with conditional heteroskedasticity, while intuition suggests that a similar result should apply, the relative performance of the tests associated with the OLS, WLS and QML estimators is not well understood. In particular, while QML has been shown to be able to generate more powerful tests than OLS, not much is known regarding the relative performance of the WLS-based test. By providing an in-depth comparison of the tests, the current paper fills this gap in the literature.

1. Introduction

An extensive literature in economics suggests that many time series are well characterized as autoregressive processes with a root near unity. Much of the formal inference apparatus used to investigate the unit root hypothesis is, however, known to suffer from low power. Take as an example the augmented Dickey–Fuller (ADF) test, which has become the workhorse of the industry, with the largest number of applications by far. The fact that this test has low power is by now so widespread that it is almost difficult to apply it without, at the same time, bringing up the power issue. As a result, the use of "confirmatory" testing strategies, in which multiple tests are used to attest the same thing, has become an important criterion for the validity of empirical results. The problem here is that, to the extent that the "other" tests are based on the same information set, this practice does not necessarily lead to higher power.

However, in many cases, there is more information to be had, and then it is not that surprising that one can do better than ADF. One source of such extraneous information that has received much attention in the literature is autoregressive conditional heteroskedasticity (ARCH). As Seo (1999) shows, while the ADF test statistic allows for ARCH in the sense that its asymptotic distribution is invariant to heteroskedasticity of this kind, it does not make use of the information contained therein. In particular, the problem is that the conditional mean equation is estimated by ordinary least squares (OLS), which does not account for the information contained in higher moments. Seo (1999) therefore proposes full quasi-maximum likelihood (QML) estimation of both the conditional mean and variance equations. Since this estimator accounts for the information contained in the ARCH, it is efficient (Ling and Li, 1998), suggesting that the QML-based tests should be relatively more powerful, a result that has been verified in several simulation studies (see, for example, Seo, 1999; Ling et al., 2003).

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Full QML estimation of the ADF test regression with ARCH errors has at least four major drawbacks, though. First, in small samples the QML estimator is known to exhibit Cauchy-like tails and therefore infinite moments, which can generate extreme outliers with misleading inference as a result (Phillips, 1994). Second, because of the poor behavior in small samples, successful maximization of the log-likelihood can be quite difficult, and in some cases borderline impossible (see, for example, Brüggemann and Lütkepohl, 2005; Herwartz and Lütkepohl, 2011). Third, unless a robust estimator of the standard error is used, the QML-based test statistics require parametric correction to account for the kurtosis of the disturbances. Fourth, while the QML-based tests have been around for quite some time now, they have not yet been implemented in any econometric software package, making them quite unattractive from an applied point of view.

In this paper we compare the QML-based tests of Seo (1999) to a test based on weighted least squares (WLS), which is similar in spirit to the ones recently considered by Andrews and Guggenberger (2009, 2012), Herwartz and Lütkepohl (2011), and Westerlund (forthcoming).[1] This test is not only computationally very convenient, requiring nothing but simple OLS operations, but is also relatively robust in the sense that it is not subject to any numerical optimization difficulties. The question is: given the poor power of the ADF test, does it pay off to use full QML or can one just as well use WLS?

Our investigation of this question consists of two parts. In the first part we derive comparable expressions for the local asymptotic power functions of the tests. The analysis of these functions yields significant insights. First, contrary to what might be expected based on the theory for the classical regression model with heteroskedastic errors, the power functions of the QML- and WLS-based tests generally do not coincide. Second, the relative power of the tests depends on the extent of the ARCH and also on the kurtosis of the errors. Third, the QML- and WLS-based tests are at least as powerful as the OLS-based ADF test, with equal power only if there is no ARCH. In the second part of the analysis we use Monte Carlo simulation to study the small-sample accuracy of the asymptotic results, and also how the performance of the tests depends on implementation issues such as lag augmentation and numerical optimization.

2. Model and assumptions

The observed variable $\{y_t\}_{t=1}^T$ is assumed to evolve according to the following data generating process (DGP):

$$y_t = \theta' d_t + s_t, \tag{1}$$

$$\phi(L)\Delta s_t = \delta s_{t-1} + \epsilon_t, \tag{2}$$

$$\delta = c\phi(1)T^{-1}, \tag{3}$$

where $\phi(L) = 1 - \sum_{j=1}^{p} \phi_j L^j$ is a polynomial in the lag operator L, s_t is the stochastic part of y_t and d_t is the deterministic part. Typical elements of d_t include a constant and a linear time trend. In this paper we consider three models: (1) $d_t = 0$ (no deterministic constant or trend terms), (2) $d_t = 1$ (a constant) and (3) $d_t = (1, t)'$ (a constant and trend). The stochastic part is modeled as a near unit root process with $c \in (-\infty, \infty)$ playing the role of a drift (or local-to-unity) parameter measuring the persistence in $\{s_t\}_{t=1}^T$. If $c = 0$, then $\delta = 0$, and therefore $\{s_t\}_{t=1}^T$ has a unit root, whereas if $c < 0$, then δ approaches 0 from below, and so $\{s_t\}_{t=1}^T$ is "locally stationary". The process can also be "locally explosive" ($c > 0$); however, since explosive behavior is rare, here we follow the usual convention in the literature and focus on the case when $c \leq 0$. The hypothesis of interest is therefore formulated as $H_0 : c = 0$ versus $H_1 : c < 0$.[2] Conditional heteroskedasticity under both H_0 and H_1 is allowed by assuming that ϵ_t follows an ARCH process of order q;

$$\epsilon_t = \sigma_t \eta_t, \tag{4}$$

$$\sigma_{\epsilon,t}^2 = \alpha_0 + \alpha(L)\epsilon_{t-1}^2, \tag{5}$$

where $\alpha(L) = \sum_{j=1}^{q} \alpha_j L^{j-1}$.

Assumption 1. *(a)* η_t is symmetric, and independently and identically distributed (iid) with $E(\eta_t) = 0$ and $E(\eta_t^2) = 1$;
(b) $E(\epsilon_t^2) = E(\sigma_{\epsilon,t}^2) = \sigma_\epsilon^2 > 0$ and $E(\epsilon_t^4) < \infty$;
(c) $\alpha_0 > 0, \alpha_1, \ldots, \alpha_q \geq 0$, and the roots of $\phi(L)$ fall outside the unit circle;
(d) $s_{-p} = \cdots = s_0 = 0$.

Remarks.

1. The ARCH assumption is made here in order to simplify the comparison between the QML and WLS estimators. In particular, while QML can accommodate generalized ARCH (GARCH) terms, the WLS estimator is based on OLS estimation of the conditional variance equation, which precludes such terms. However, as long as the purpose is just to estimate

[1] While Andrews and Guggenberger (2009, 2012), Westerlund (forthcoming) consider tests for a unit root, Herwartz and Lütkepohl (2011) consider tests for cointegration.

[2] Thus, while the results provided in this paper hold for all $c \in (-\infty, \infty)$, in the interpretation of the test outcome we follow the usual convention and assume that $c \leq 0$, such that the tests can be set up as one-sided.