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The gamma-normal distribution: Properties and applications

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ABSTRACT

In this paper, some properties of gamma-X family are discussed and a member of the family, the gamma-normal distribution, is studied in detail. The limiting behaviors, moments, mean deviations, dispersion, and Shannon entropy for the gamma-normal distribution are provided. Bounds for the non-central moments are obtained. The method of maximum likelihood estimation is proposed for estimating the parameters of the gamma-normal distribution. Two real data sets are used to illustrate the applications of the gamma-normal distribution.

1. Introduction

There are several methods to generate continuous distributions. Many of these methods are discussed in the book by Johnson et al. (1994, Chapter 12). Since the publication of the book, new methods continue to appear in the literature. Eugene et al. (2002) introduced the beta-generated class of distributions and pointed out that the distributions of order statistics are special cases of beta-generated distributions. Jones (2004) studied some properties of beta-generated distributions. Many beta-generated distributions have been studied (e.g., Famoye et al. (2004, 2005), Nadarajah and Kotz (2006), Akinsete et al. (2008), Barreto-Souza et al. (2010), and Alshawarbeh et al. (2012)). The method leading to beta-generated distributions was extended by using a generalized beta distribution as the generator (Jones, 2009; Cordeiro and de Castro, 2011). Ferreira and Steel (2006) used inverse probability integral transformation method to generate skewed distributions, which include the skewed normal family introduced by Azzalini (1985, 2005) as a special class. Recently, Alzaatreh et al. (2013b) developed a new method to generate family of distributions and called it the T–X family of distributions. For a review of methods for generating univariate continuous distributions, one may refer to Lee et al. (2013). This article has two purposes. First, we take T as a gamma random variable, X as any continuous random variable and study some general properties of the gamma-X family. Second, we study in detail the gamma-normal distribution, which is a member of the gamma-X family.

Let $F(x)$ be the cumulative distribution function (CDF) of any random variable X and $r(t)$ be the probability density function (PDF) of a random variable T defined on $[0, \infty)$. The CDF of the T–X family of distributions defined by Alzaatreh et al. (2013b) is given by

$$G(x) = \int_0^{-\log(1-F(x))} r(t)dt. \tag{1.1}$$

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Alzaatreh et al. (2013b) named the family of distributions defined in (1.1) the 'Transformed-Transformer' family (or T–X family). When X is a continuous random variable, the probability density function of the T–X family is

$$g(x) = \frac{f(x)}{1-F(x)} r\left(-\log\left(1-F(x)\right)\right) = h(x)\, r\left(H(x)\right). \tag{1.2}$$

Thus, the family of distributions defined in (1.2) can be viewed as a family of distributions arising from hazard functions. If a random variable T follows the gamma distribution with parameters α and β, $r(t) = \left(\beta^{\alpha}\,\Gamma(\alpha)\right)^{-1} t^{\alpha-1} e^{-t/\beta}$, $t \geq 0$. The definition in (1.2) leads to the gamma-X family with the PDF

$$g(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} f(x) \left(-\log\left(1-F(x)\right)\right)^{\alpha-1} \left(1-F(x)\right)^{1/\beta-1}. \tag{1.3}$$

The CDF of the gamma-X distribution in (1.3) can be written as

$$G(x) = \frac{\gamma\{\alpha, -\log(1-F(x))/\beta\}}{\Gamma(\alpha)}, \tag{1.4}$$

where $\gamma(\alpha, t) = \int_0^t u^{\alpha-1} e^{-u} du$ is the incomplete gamma function.

The Weibull-X family along with a member, Weibull–Pareto distribution, was studied by Alzaatreh et al. (2013a). By using geometric distribution as the distribution of the random variable X in the T–X family, Alzaatreh et al. (2012) derived the family of discrete analogues of continuous random variables. In Section 2, we provide some properties of the gamma-X family. In the remaining sections, the gamma-normal distribution is studied in detail. In Section 3, we study some properties of the gamma-normal distribution including unimodality, quantile function and Shannon's entropy. Series representation and bounds for the non-central moments of the gamma-normal distribution are studied in Section 3. Section 4 deals with the method of maximum likelihood for estimating the parameters of the gamma-normal distribution. Applications of the distribution to real data sets are provided in Section 5.

2. The gamma-X family

In this section, some general properties of the gamma-X in (1.3) are discussed. The following are some special cases of the gamma-X family:

1. When $\alpha = 1$, the gamma-X family in (1.3) reduces to $g(x) = \beta^{-1} f(x) \left(1-F(x)\right)^{1/\beta-1}$, which is the distribution of the first order statistic from a random sample of size $n\,(=\beta^{-1})$ with PDF $f(x)$.
2. Arnold et al. (1998, Chapter 2) defined the CDF of the upper record value U_n as

$$G_U(u) = P(U_n \leq u) = \int_0^{-\log(1-F(u))} [w^n e^{-w}/n!] dw. \tag{2.1}$$

The corresponding PDF is

$$g_U(u) = f(u)[-\log(1-F(u))]^n/n! \tag{2.2}$$

If the "transformed" random variable T in (1.1) follows a gamma distribution with $\alpha = n+1$ and $\beta = 1$, the distribution in (1.1) reduces to (2.1) and the PDF in (1.3) reduces to the PDF in (2.2). Thus, the family of upper record value distributions is a special case of gamma-X family.

3. Gamma-X distribution in (1.3) can be written as

$$g(x) = \frac{f(x) \left(-\log[1-F(x)]\right)^{\alpha-1} \lambda^{\alpha} \left(1-F(x)\right)^{\lambda-1}}{\Gamma(\alpha)} = q(x)\lambda^{\alpha}\left(1-F(x)\right)^{\lambda-1} = q(x)w(x). \tag{2.3}$$

The function $q(x)$ is the gamma-generated PDF by Zografos and Balakrishnan (2009), which is given by setting $\beta = 1$ in (1.3). The function $w(x)$ in (2.3) can be viewed as a weight function. Thus, the gamma-X PDF is the gamma-generated distribution $q(x)$ weighted by $w(x)$. The weight $w(x)$ is a function of the survival function of the random variable X with CDF $F(x)$. When $\lambda > 1$, $w(x)$ is a decreasing function of x and when $\lambda < 1$, $w(x)$ is an increasing function of x.

4. Similar to the result in (1.3) for the distributions of the upper record values, one can derive a family of distributions arising from distribution of the lower record values by using the CDF

$$G_L(x) = 1 - \int_0^{-\log(F(x))} \left(\beta^{\alpha}\,\Gamma(\alpha)\right)^{-1} t^{\alpha-1} e^{-t/\beta} dt, \tag{2.4}$$

and the corresponding PDF

$$g_L(x) = [\Gamma(\alpha)\beta^{\alpha}]^{-1} f(x) \left(-\log\left(F(x)\right)\right)^{\alpha-1} \left(F(x)\right)^{1/\beta-1}. \tag{2.5}$$

When $\alpha = n+1$ and $\beta = 1$, the density in (2.5) is the density of the nth lower record value.