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Multivariate topology simplification

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ABSTRACT

Topological simplification of scalar and vector fields is well-established as an effective method for analysing and visualising complex data sets. For multivariate (alternatively, multi-field) data, topological analysis requires simultaneous advances both mathematically and computationally. We propose a robust multivariate topology simplification method based on “lip”-pruning from the Reeb space. Mathematically, we show that the projection of the Jacobi set of multivariate data into the Reeb space produces a Jacobi structure that separates the Reeb space into simple components. We also show that the dual graph of these components gives rise to a Reeb skeleton that has properties similar to the scalar contour tree and Reeb graph, for topologically simple domains. We then introduce a range measure to give a scaling-invariant total ordering of the components or features that can be used for simplification. Computationally, we show how to compute Jacobi structure, Reeb skeleton, range and geometric measures in the Joint Contour Net (an approximation of the Reeb space) and that these can be used for visualisation similar to the contour tree or Reeb graph.

1. Introduction

Scientific data is often complex in nature and difficult to visualise. As a result, analytic tools have become increasingly prominent in scientific visualisation, and in particular topological analysis. While earlier work dealt primarily with scalar data [1–3], multivariate topological analysis in the form of the Reeb space [4,5] has started to become feasible using a quantised approximation called the Joint Contour Net (JCN) [6].

Prior experience in scalar and vector topology shows that simplification of topological structures is required, as real data sets are often noisy and complex. Although most of the work required is practical and algorithmic in nature, mathematical formalisms are also needed, in this case based on fiber analysis, in the same way that Reeb graphs and contour trees rely on Morse theory. This paper therefore:

1. Clarifies relationships between the Reeb space of a multivariate map f, the Jacobi set of f, and fiber topology;

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2. Introduces the *Jacobi structure* in the Reeb space that decomposes the Reeb space into *regular* and *singular* components equivalent to edges and vertices in the Reeb graph, then reduces it further to a *Reeb skeleton*;
3. Proves that Reeb spaces for topologically simple domains have simple structures with properties analogous to properties of the contour tree, allowing *lip-pruning* based simplification;
4. Introduces the *range measure* and other geometric measures for a total ordering of regular components of the Reeb space;
5. Describes an algorithm that extracts the Jacobi structure from the Joint Contour Net using a *Multi-Dimensional Reeb Graph* (MDRG) and computes the Reeb skeleton, and
6. Simplifies the Reeb skeleton and the corresponding Reeb space computing the range and other geometric measures using the JCN.

To clarify the relationships between the newly introduced data-structures in the current paper, note that the JCN is an approximation of the Reeb space. We compute a MDRG from the JCN. The critical nodes of the MDRG form the Jacobi structure of the JCN. The Jacobi structure then separates the JCN into regular and singular components. The dual graph of such components gives a Reeb skeleton which is used in the multivariate topology simplification.

As a result, much of this paper addresses the theoretical machinery for simplification of the Reeb space and its approximation, the Joint Contour Net. Section 2 reviews relevant background material on simplification, followed by a more detailed review of the fiber topology, Jacobi set and Reeb space in Section 3. Section 4 provides theoretical analysis and results needed for the lip-simplification of the Reeb space. For simple domains, the Reeb space can have detachable (lip) components: this is used in Section 5 to generalise leaf-pruning simplification from the contour tree to the Reeb space. Once this has been done, we introduce a range persistence and other geometric measures to govern the simplification process.

In Section 6, we give an algorithm for simplifying the Joint Contour Net. We start by building a hierarchical structure called the Multi-Dimensional Reeb Graph that captures the Jacobi structure of the Joint Contour Net, and then show how to reduce the JCN to a Reeb skeleton – a graph with properties similar to a contour tree. In Section 7, we illustrate these reductions first with analytic data where the correct solution is known *a priori*, then for a real data from the nuclear physics. As part of this, we provide performance figures and other implementation details in Section 7, then draw conclusions and lay out a road map for further work in Section 8.

2. Previous work

Topology-based simplification aims to reduce the topological complexity of the underlying data. There are different ways to measure such topological complexity depending on the nature of the underlying data. Here we mention some well-known approaches from the literature for measuring the topological complexity and their simplification procedure.

2.1. Scalar field simplification

The topological complexity of the scalar field data is measured in terms of the number of critical points and their connectivities – captured by its Reeb graph or contour tree. Another way to capture the topological complexity of the scalar field is by computing the Morse–Smale complex of the corresponding gradient field. Therefore, the topological simplification in this case is driven by reducing the number of critical points via simplification of the Reeb graph, contour tree or the Morse–Smale complex. Carr et al. [2] describe a method for associating local geometric measures such as the surface area and the contained volume of contours with the contour tree and then simplifying the contour tree by suppressing the minor *topological features* of the data. Note that a feature is any prominent or distinctive part or quality that characterises the data and topological features captures the topological phenomena of the underlying data. Wood et al. [7] give a Reeb graph based simplification strategy for removing the excess topology created by unwanted handles in an isosurface using a measure for computing the handle-size in the isosurface and associating them with the loops of the Reeb graph. Gyulassy et al. [8] describe a technique for simplifying a three-dimensional scalar field by repeatedly removing pair of critical points from the Morse–Smale complex of its gradient field, by repeated application of a critical-point simplification operation. Mathematically, the simplification of "lips" proposed in this paper is a direct generalization of this idea (for scalar fields) to multi-fields. Luo et al. [9] describe a method for computing and simplifying gradients and critical points of a function from a point cloud. Tierny et al. [3] present a combinatorial algorithm for simplifying the topology of a scalar field on a surface by approximating with a simpler scalar field having a subset of critical points of the given field, while guaranteeing a small error distance between the fields.

The topological complexity of a point cloud data can be measured by its homology. For a point cloud data in $\mathbb{R}^3$ this is expressed by the topological invariants, such as the Betti numbers corresponding to a simplicial complex of the point cloud – denoted by β_0 (number of connected components), β_1 (number of tunnels or 1-dimensional holes) and β_2 (number of voids or 2-dimensional holes). The i-th Betti number represents the rank of the i-th homology group ($i = 0, 1, 2$). Edelsbrunner et al. [10] introduce the idea of persistence for the topological simplification of a point cloud by reducing the Betti numbers using a filtration technique. Cohen-Steiner et al. [11] extend the persistence diagram for scalar functions on topological spaces and analyze its stability.