Computational Geometry 58 (2016) 25–33

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

Drawing graphs with vertices and edges in convex position

Ignacio García-Marco [a,1], Kolja Knauer [b,*,2]

[a] LIP, ENS Lyon – CNRS – UCBL – INRIA, Université de Lyon, UMR 5668, Lyon, France
[b] Aix-Marseille Université, CNRS, LIF UMR 7279, Marseille, France

ARTICLE INFO

Article history:
Received 7 April 2016
Accepted 9 June 2016
Available online 22 June 2016

Keywords:
Strong convex dimension
Minkowski sum
Planar point sets

ABSTRACT

A graph has strong convex dimension 2 if it admits a straight-line drawing in the plane such that its vertices form a convex set and the midpoints of its edges also constitute a convex set. Halman, Onn, and Rothblum conjectured that graphs of strong convex dimension 2 are planar and therefore have at most $3n-6$ edges. We prove that all such graphs have indeed at most $2n-3$ edges, while on the other hand we present an infinite family of non-planar graphs of strong convex dimension 2. We give lower bounds on the maximum number of edges a graph of strong convex dimension 2 can have and discuss several natural variants of this graph class. Furthermore, we apply our methods to obtain new results about large convex sets in Minkowski sums of planar point sets – a topic that has been of interest in recent years.

1. Introduction

A point set $X \subseteq \mathbb{R}^2$ is *(strictly) convex* if every point in X is a vertex of the convex hull of X. A point set X is said to be *weakly convex* if X lies on the boundary of its convex hull. A *drawing* of a graph G is a mapping $f : V(G) \to \mathbb{R}^2$ such that edges are straight line segments connecting vertices and neither midpoints of edges, nor vertices, nor midpoints and vertices coincide. Through most of the paper we will not distinguish between (the elements of) a graph and their drawings.

For $i, j \in \{s, w, a\}$ we define $\mathcal{G}_i^j$ as the class of graphs admitting a drawing such that the set of vertices is $\begin{cases} \text{strictly convex} & \text{if } i = s \\ \text{weakly convex} & \text{if } i = w \\ \text{arbitrary} & \text{if } i = a \end{cases}$ and the midpoints of edges constitute a $\begin{cases} \text{strictly convex} & \text{if } j = s \\ \text{weakly convex} & \text{if } j = w \\ \text{arbitrary} & \text{if } j = a \end{cases}$ set. Further, we define $g_i^j(n)$ to be the maximum number of edges an n-vertex graph in $\mathcal{G}_i^j$ can have.

Clearly, all $\mathcal{G}_i^j$ are closed under taking subgraphs and $\mathcal{G}_s^a = \mathcal{G}_w^a = \mathcal{G}_a^a$ is the class of all graphs.

Previous results and related problems: Motivated by a special class of convex optimization problems [5], Halman, Onn, and Rothblum [4] studied drawings of graphs in $\mathbb{R}^d$ with similar constraints as described above. In particular, in their language a graph has convex dimension 2 if and only if it is in $\mathcal{G}_a^s$ and strong convex dimension 2 if and only if it is in $\mathcal{G}_s^s$. They show

* Corresponding author.
E-mail addresses: ignacio.garcia-marco@ens-lyon.fr (I. García-Marco), kolja.knauer@lif.univ-mrs.fr (K. Knauer).

1 Supported by ANR project CompA (project number: ANR-13-BS02-0001-01).
2 Supported by ANR EGOS grant ANR-12-JS02-002-01 and PEPS grant EROS.

http://dx.doi.org/10.1016/j.comgeo.2016.06.002
0925-7721/

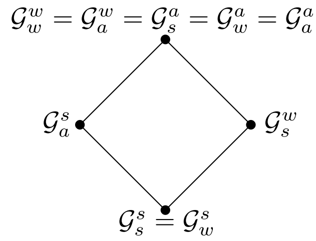

Fig. 1. Inclusions and identities among the classes $\mathcal{G}_i^j$.

that all trees and cycles are in $\mathcal{G}_s^s$, while $K_4 \in \mathcal{G}_a^s \setminus \mathcal{G}_s^s$ and $K_{2,3} \notin \mathcal{G}_a^s$. Moreover, they show that $n \le g_s^s(n) \le 5n-8$. Finally, they conjecture that all graphs in $\mathcal{G}_s^s$ are planar and thus $g_s^s(n) \le 3n-6$.

The problem of computing or bounding $g_a^s(n)$ and $g_s^s(n)$ was rephrased and generalized in the setting of convex subsets of Minkowski sums of planar point sets by Eisenbrand et al. [2] and then regarded as a problem of computational geometry in its own right. We introduce this setting and give an overview of known results before explaining its relation to the original graph drawing problem.

Given two point sets $A, B \subseteq \mathbb{R}^d$ their *Minkowski sum* $A+B$ is defined as $\{a+b \mid a \in A, b \in B\} \subseteq \mathbb{R}^d$. We define $M(m,n)$ as the largest cardinality of a convex set $X \subseteq A+B$, for A and B planar point sets with $|A| = m$ and $|B| = n$. In [2] it was shown that $M(m,n) \in O(m^{2/3}n^{2/3}+m+n)$. This upper bound was complemented by Bílka et al. [1] with an asymptotically matching lower bound, even under the assumption that A itself is convex, i.e., $M(m,n) \in \Theta(m^{2/3}n^{2/3}+m+n)$. Notably, the lower bound works also for the case $A = B$ non-convex, as shown by Swanepoel and Valtr [6, Proposition 4]. In [7] Tiwary gives an upper bound of $O((m+n)\log(m+n))$ for the largest cardinality of a convex set $X \subseteq A+B$, for A and B planar convex point sets with $|A| = m$ and $|B| = n$. Determining the asymptotics in this case remains an open question.

As first observed in [2], the graph drawing problem of Halman et al. is related to the largest cardinality of a convex set $X \subset A+A$, for A some planar point set. In fact, from X and A one can deduce a graph $G \in \mathcal{G}_a^s$ on vertex set A, with an edge aa' for all $a \neq a'$ with $a+a' \in X$. The midpoint of the edge aa' then just is $\frac{1}{2}(a+a') \in \frac{1}{2}X \subset \frac{1}{2}A + \frac{1}{2}A$. Conversely, from any $G \in \mathcal{G}_a^s$ one can construct X and A as desired. The only trade-off in this translation are the pairs of the form aa, which are not taken into account by the graph-model, because they correspond to vertices. Hence, they do not play a role from the purely asymptotic point of view. Thus, the results of [2,1,6] yield $g_a^s(n) = \Theta(n^{4/3})$. Conversely, the bounds for $g_s^s(n)$ obtained in [4] give that the largest cardinality of a convex set $X \subseteq A+A$, for A a planar convex point set with $|A| = n$ is in $\Theta(n)$.

Our results: In this paper we study the set of graph classes defined in the introduction. We extend the list of properties of point sets considered in earlier works with *weak* convexity. We completely determine the inclusion relations on the resulting classes. We prove that $\mathcal{G}_s^s$ contains non-planar graphs, which disproves a conjecture of Halman et al. [4], and that $\mathcal{G}_s^w$ contains cubic graphs, while we believe is false for $\mathcal{G}_s^s$. We give new bounds for the parameters $g_i^j(n)$: we show that $g_s^w(n) = 2n-3$, which is an upper bound for $g_s^s(n)$ and therefore improves the upper bound of $3n-6$ conjectured by Halman et al. [4]. Furthermore we show that $\lfloor \frac{3}{2}(n-1) \rfloor \le g_s^s(n)$.

For the relation with Minkowski sums we show that the largest cardinality of a weakly convex set $X \subseteq A+A$, for A some convex planar point set of $|A| = n$, is $2n$ and of a strictly convex set is between $\frac{3}{2}n$ and $2n-2$.

The results for weak convexity are the first non-trivial precise formulas in this area.

A preliminary version of this paper has been published in conference proceedings [3].

2. Graph drawings

Given a graph G drawn in the plane with straight line segments as edges, we denote by P_V the convex hull of its set of vertices and by P_E the convex hull of the set of midpoints of its edges. Clearly, unless $V = \emptyset$, P_E is strictly contained in P_V.

2.1. Inclusions of classes

We show that most of the classes defined in the introduction coincide and determine the exact set of inclusions among the remaining classes.

Theorem 1. *We have $\mathcal{G}_s^s = \mathcal{G}_w^s \subsetneq \mathcal{G}_s^w \subsetneq \mathcal{G}_w^w = \mathcal{G}_a^w = \mathcal{G}_s^a = \mathcal{G}_w^a = \mathcal{G}_a^a$ and $\mathcal{G}_s^s \subsetneq \mathcal{G}_a^s \subsetneq \mathcal{G}_w^w$. Moreover, there is no inclusion relationship between $\mathcal{G}_a^s$ and $\mathcal{G}_s^w$. See Fig. 1 for an illustration.*

Proof. Let us begin by proving that $\mathcal{G}_s^s = \mathcal{G}_w^s$, the inclusion $\mathcal{G}_s^s \subseteq \mathcal{G}_w^s$ is obvious. Take $G \in \mathcal{G}_w^s$ drawn in the required way. Since the midpoints of the edges form a convex set, there exists $\delta > 0$ such that moving every vertex by at most $< \delta$ in any direction, the set of midpoints of the edges remains strictly convex. More precisely, whenever there are vertices $z_1, \ldots, z_k$ in the interior of the segment connecting two vertices x, y, we perform the following steps, see Fig. 2: