Computational Geometry 58 (2017) 60–69

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

Affine invariants of generalized polygons and matching under affine transformations

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ARTICLE INFO

Article history:
Received 16 December 2014
Received in revised form 9 August 2015
Accepted 7 June 2016
Available online 23 June 2016

Keywords:
Indexed affine matching
Fourier descriptor
Self-intersecting polygon
Pseudo-hyperbolic distance

ABSTRACT

A generalized polygon is an ordered set of vertices. This notion generalizes the concept of the boundary of a polygonal shape because self-intersections are allowed. In this paper we study the problem of matching generalized polygons under affine transformations. Our approach is based on invariants. Firstly we associate an ordered set of complex numbers with each polygon and construct a collection of complex scalar functions on the space of plane polygons. These invariant functions are defined as quotients of the so-called Fourier descriptors, also known as discrete Fourier transforms.

Each one of these functions is invariant under similarity transformations; that is, the function associates the same complex number to similar polygons. Moreover, if two polygons are affine related (one of them is the image of the other under an affine transformation), the pseudo-hyperbolic distance between their associated values is a constant that depends only on the affine transformation involved, but independent of the polygons.

More formally, given a collection $\{Z_1, Z_2, \ldots, Z_m\}$ of n-sided polygons in the plane and a query polygon W, we give algorithms to find all Z_ℓ such that $f(Z_\ell) = W + \Delta W$, where f is an unknown affine transformation and $\Delta W = (\Delta w_1, \ldots, \Delta w_n)$ with $|\Delta w_k| \leq \rho$, where ρ is certain tolerance.

1. Introduction

Informally speaking, a shape is the contour of an object in two dimensions, or a curve describing the boundary of an object. However, objects may have holes and the notion of contour needs to be more precise to avoid misunderstandings. Usually a shape is considered as a compact set in the plane with a connected boundary; the boundary of the shape is a Jordan curve (i.e. non-self-intersecting) in the plane. Shapes appear in many applications fields like computer-aided design, computer-aided manufacturing, computer vision [1], medical imaging [2] and even archaeology [3]. Shape analysis deals with the concept of matching shapes. The definition of matching changes slightly with the application field, and ranges from congruence transformations (where the shape could be rotated, translated or reflected, without being resized), similarity transformations [1] (include resizing to the previous set of transformations), affine transformations (include non-uniform resizing and shearing in addition to similarity transformations), projective transformations [4], to Riemannian isometries [5] (for curved surfaces) and conformal mappings [6], or more general transformations. Matching could also include partial

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http://dx.doi.org/10.1016/j.comgeo.2016.06.003
0925-7721/

matching, where only a portion of the shape have a match. As a rule of thumb, the more general the transformation is more difficult to find a fast algorithm to find the best match. For an arbitrary transformation (e.g., elastic matching), the problem is NP-complete [7].

There are a number of different ways to represent the boundary of a shape. In this paper we assume that each shape boundary is represented as an n-sided closed polygonal.

There are efficient algorithms to decide if two polygons are affine related. An algorithm presented in [8] determines whether two polygons are similar in $O(MN\log(MN))$ time and $O(M+N)$ space, with M and N the number of vertices in the respective polygons. They use turn functions, which are discrete versions of the curvature function of a curve. Those are piecewise constant functions increasing with left-hand turns and decreasing with right-hand turns measured from a starting axis. An efficient partial matching is possible even in the presence of noise. Several books and surveys are devoted to these problems [9,10,1].

1.1. The problem: indexed polygon matching

We shall identify points (x, y) in the plane with corresponding complex numbers $z = x + iy \in \mathbb{C}$, where i is the imaginary unity, which satisfies $i^2 = -1$. A *(generalized) polygon* in the plane is represented as an ordered set of points, or complex numbers, where the *order specifies consecutive vertices*. *Self-intersections are allowed*. Notice that different labels for the n-sided polygon $(z_1, z_2, \dots, z_{n-1}, z_n)$ are the cyclic shifts $(z_2, z_3, \dots, z_n, z_1)$, $(z_3, z_4, \dots, z_1, z_2)$, $\dots$, $(z_n, z_1, \dots, z_{n-2}, z_{n-1})$ depending on the vertex from which starts listing the vertices. The same polygon may also be labeled in reversed order, namely $(z_n, z_{n-1}, \dots, z_2, z_1)$, $(z_1, z_n, \dots, z_3, z_2), \dots, (z_{n-1}, z_{n-2} \dots, z_1, z_n)$. We call a *cyclic relabeling* to any of those different labels of the same polygon.

An affine transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a composition of a lineal isomorphism and a translation, that is

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix},$$

where $ad - bc \neq 0$. By setting

$$\alpha = \frac{a+d}{2} + i\frac{c-b}{2}, \quad \beta = \frac{a-d}{2} + i\frac{c+b}{2}, \quad \gamma = r + is,$$

the affine transformation f can be written in terms of sums and products of complex numbers in the form

$$f(z) = \alpha z + \beta \bar{z} + \gamma.$$

Here $\bar{z}$ stands for the complex conjugated of z.

Given the polygons $Z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ and $W = (w_1, w_2, \dots, w_n) \in \mathbb{C}^n$, our problem consists of determining if there exists an affine transformation f such that $Z = (f(w_{\mathfrak{p}(1)}), \dots, f(w_{\mathfrak{p}(n)}))$, for some *cyclic permutation* $\mathfrak{p} : \{1, \dots, n\} \to \{1, \dots, n\}$.

It is convenient to formulate the problem of indexed matching. Here a collection $Z_1, Z_2, \dots, Z_m$ of polygons is preprocessed and stored, and a query polygon W is presented to the system. The outcome will be all the Z_ℓ in the collection which are affine images of any cyclic relabeling of W.

1.2. Summary of results

Given a collection $\{Z_1, Z_2, \dots, Z_m\}$ of n-sided polygons and an n-sided query polygon W, we give algorithms to solve the following problems under the *REAL RAM* model of computation:

1. Finding exact matches of W, including cyclic relabeling of W, under an unknown similarity or a known affine transformation, in time independent of m (the size of the collection).
2. Finding *all* the approximate matches of W under unknown noisy similarities with cyclic relabeling in time $O(Rn^2 \log m)$, with R the number of points in a circular range query of a certain radius, which is discussed later in the paper. Without cyclic relabeling the time to find all the matches is $O(Rn\log m)$.
3. Finding exact matches, included cyclic relabeling, under unknown affine transformations in $O(mn^2)$ time.
4. Finding *all* the approximate matches, also including cyclic relabeling, under unknown affine transformations in time $O(m(R + n^2))$, with R the number of matches of certain indicator functions, derived from our invariants, discussed later on the paper.

For the matching candidates we know the correspondence with respect to the query polygon. In the presence of noise it is possible to find the best match, in the least square sense, using affine regression in $O(n)$ operations.

Our approach to n-sided polygon matching under similarity transformations involves the construction of $\lfloor (n-1)/2 \rfloor$ complex scalar functions $\varphi_j : \mathbb{C}^n \to \mathbb{C}$. Functions φ_j are useful to recognize polygons because theoretically the probability that two random polygons are mapped to the same complex number is zero (Remark 2.7). Each φ_j assigns the same value