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Thickness and colorability of geometric graphs ☆

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ABSTRACT

The geometric thickness of a graph G is the smallest integer t such that there exist a straight-line drawing Γ of G and a partition of its straight-line edges into t subsets, where each subset induces a planar drawing in Γ. Over a decade ago, Hutchinson, Shermer, and Vince proved that any n-vertex graph with geometric thickness two can have at most $6n-18$ edges, and for every $n \geq 8$ they constructed a geometric thickness-two graph with $6n-20$ edges. In this paper, we construct geometric thickness-two graphs with $6n-19$ edges for every $n \geq 9$, which improves the previously known $6n-20$ lower bound. We then construct a thickness-two graph with 10 vertices that has geometric thickness three, and prove that the problem of recognizing geometric thickness-two graphs is NP-hard, answering two questions posed by Dillencourt, Eppstein and Hirschberg. Finally, we prove the NP-hardness of coloring graphs of geometric thickness t with $4t-1$ colors, which strengthens a result of McGrae and Zito, when $t=2$.

1. Introduction

The *thickness* $\theta(G)$ of a graph G is the smallest integer t such that the edges of G can be partitioned into t subsets, where each subset induces a planar graph. Since 1963, when Tutte [2] first formally introduced the notion of graph thickness, this property of graphs has been extensively studied for its interest from both the theoretical [3–5] and practical point of view [6,7]. A wide range of applications in circuit layout design and network visualization, have motivated the examination of thickness in the geometric setting [5,8,9]. The *geometric thickness* $\overline{\theta}(G)$ of a graph G is the smallest integer t such that there exist a *straight-line drawing* (i.e., a drawing on the Euclidean plane, where every vertex is drawn as a point and every edge is drawn as a straight line segment) Γ of G and a partition of its straight-line edges into t subsets, where each subset induces a planar drawing in Γ. If $t=2$, then G is called a *geometric thickness-two graph* (or, a doubly-linear graph [9]), and Γ is called a *geometric thickness-two representation* of G. While thickness does not impose any restriction on the placement of the vertices in each planar layer, geometric thickness forces the same vertices in different planar layers to share a fixed point in the plane. Eppstein [8] clearly established this difference by constructing thickness-three graphs that have arbitrarily large geometric thickness.

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1.1. Structural properties

Geometric thickness has been broadly examined on several classes of graphs, e.g., complete graphs [5], bounded-degree graphs [10,11,8], and graphs with bounded treewidth [12,13]. Hutchinson, Shermer, and Vince [9] examined properties of graphs with geometric thickness two. They proved that these graphs can have at most $6n - 18$ edges, and for every $n \geq 8$ they constructed a geometric thickness-two graph with $6n - 20$ edges. The graphs that gave the $6n - 20$ lower bound were *rectangle visibility graphs*, i.e., these graphs can be represented such that the vertices are axis-aligned rectangles on the plane with adjacency determined by the horizontal and vertical visibility. Hutchinson et al. [9] proved that a rectangle visibility graph can have at most $6n-20$ edges, therefore, any geometric thickness-two graph with more than $6n-20$ edges (if exists) cannot be a rectangle visibility graph. Even after several attempts [5,11] to understand the structural properties of geometric thickness-two graphs, the question whether there exists a geometric thickness two graph with $6n-18$ edges remained open for over a decade. Answering this question is quite challenging since although one can generate many thickness-two graphs with $6n - 18$ or $6n - 19$ edges, no efficient algorithm is known that can determine the geometric thickness of such a graph. However, by examining the point configurations that are likely to support geometric thickness-two graphs with large numbers of edges, we have been able to construct geometric thickness-two graphs with $6n - 19$ edges, which improves the previously known $6n - 20$ lower bound on the maximum number of edges that a graph with geometric thickness two can have. In Section 2 we have shown that the K_9 minus an edge is a thickness-two graph, which has $6n - 19$ edges. We then show that thickness-two graphs that do not contain K_9 minus an edge may also have large number of edges.

Theorem 1. *For each $n \geq 9$, there exists a geometric thickness-two graph with n vertices and $6n - 19$ edges that contains K_9 minus an edge as a subgraph. For each $n \geq 11$, there exists a geometric thickness-two graph with $6n - 19$ edges that does not contain K_8.*

1.2. Recognition

Although thickness is known for all complete graphs [3] and complete bipartite graphs [4], geometric thickness for these graph classes is not completely characterized. Dillencourt, Eppstein and Hirschberg [5] proved an $\lceil n/4 \rceil$ upper bound on the geometric thickness of K_n, giving a nice construction for drawing graphs with $\lceil n/4 \rceil$ planar layers. They also gave a lower bound on the geometric thickness of complete graphs that matches the upper bound for several smaller values of n. Their bounds show that the geometric thickness of K_{15} is greater than its thickness, i.e., $\overline{\theta}(K_{15}) = 4 > \theta(K_{15}) = 3$, which settles the conjecture of [14] on the relation between thickness and geometric thickness. Since the exact values of $\overline{\theta}(K_{13})$ and $\overline{\theta}(K_{14})$ are still unknown, Dillencourt et al. [5] hoped that the relation $\overline{\theta}(G) > \theta(G)$ could be established with a graph of smaller cardinality. In Section 3 we prove that the smallest such graph contains 10 vertices.

Theorem 2. *For every $n \leq 9$ and every graph G on n vertices, $\overline{\theta}(G) = \theta(G)$. For every graph $n > 10$, there exists a graph G' on n vertices such that $\overline{\theta}(G) > \theta(G)$.*

Since determining the thickness of an arbitrary graph is NP-hard [6], Dillencourt et al. [5] suspected that determining geometric thickness might be also NP-hard, and mentioned it as an open problem. The hardness proof of Mansfield [6] relies heavily on the fact that $\theta(K_{6,8}) = 2$. Dillencourt et al. [5] mentioned that this proof cannot be immediately adapted to prove the hardness of the problem of recognizing geometric thickness-two graphs by showing that $\overline{\theta}(K_{6,8}) = 3$. This complexity question has been repeated several times in the literature [12,8] since 2000, and also appeared as one of the selected open questions in the 11th International Symposium on Graph Drawing (GD 2003) [15]. In Section 4 we settle the question by proving the problem to be NP-hard.

Theorem 3. *It is NP-hard to determine whether the geometric thickness of an arbitrary graph is at most two.*

1.3. Colorability

As a natural generalization of the well-known Four Color Theorem for planar graphs [16], a long-standing open problem in graph theory is to determine the relation between thickness and colorability [17,18]. For every $t \geq 3$, the best known lower bound on the chromatic number of the graphs with thickness t is $6t - 2$, which can be achieved by the largest complete graph of thickness t. On the other hand, every graph with thickness t is $(6t)$-colorable [17]. Recently, McGrae and Zito [19] examined a variation of this problem that given a planar graph and a partition of its vertices into subsets of at most r vertices, asks to assign a color (from a set of s colors) to each subset such that two adjacent vertices in different subsets receive different colors. They proved that the problem is NP-complete when $r = 2$ (respectively, $r > 2$) and $s \leq 6$ (respectively, $s \leq 6r - 4$) colors. In Section 5 we prove the NP-hardness of coloring geometric thickness-t graphs with $4t - 1$ colors. As a corollary, we strengthen the result of McGrae and Zito [19] that coloring thickness-$(t = r = 2)$ graphs with 6 colors is NP-hard. Our hardness result is particularly interesting since no geometric thickness-t graph with chromatic number more than $4t$ is known.