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A duality transform for constructing small grid embeddings of 3d polytopes ☆

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ABSTRACT

We study the problem of how to obtain an integer realization of a 3d polytope when an integer realization of its dual polytope is given. We focus on grid embeddings with small coordinates and develop novel techniques based on Colin de Verdière matrices and the Maxwell–Cremona lifting method.

We show that every truncated 3d polytope with n vertices can be realized on a grid of size $O(n^{9\log 6+1})$. Moreover, for every simplicial 3d polytope with n vertices with maximal vertex degree Δ and vertices placed on an $L \times L \times L$ grid, a dual polytope can be realized on an integer grid of size $O(nL^{3\Delta+9})$. This implies that for a class $\mathcal{C}$ of simplicial 3d polytopes with bounded vertex degree and polynomial size grid embedding, the dual polytopes of $\mathcal{C}$ can be realized on a polynomial size grid as well.

1. Introduction

By Steinitz's theorem the graphs of convex 3d polytopes[1] are exactly the planar 3-connected graphs [16]. Several methods are known for realizing a planar 3-connected graph G as a polytope with graph G on the grid [4,8,13,12,14,15]. It is challenging to find algorithms that produce polytopes with small integer coordinates. Having a realization with small grid size is a desirable feature, since then the polytope can be stored and processed efficiently. Moreover, grid embeddings imply good vertex and edge resolution. Hence, they produce "readable" drawings.

In 2d, every planar 3-connected graph with n vertices can be drawn with straight-line edges on an $O(n) \times O(n)$ grid without crossings [5], and a drawing with convex faces can be realized on an $O(n^{3/2} \times n^{3/2})$ grid [2]. For the realization as a polytope the currently best algorithm guarantees an integer embedding with coordinates of size at most $O(147.7^n)$ [3,13]. The current best lower bound is $\Omega(n^{3/2})$ [1]. Closing this gap is an intriguing open problem in lower dimensional polytope theory.

Recently, progress has been made for a special class of 3d polytopes, the so-called *stacked polytopes*. A *stacking operation* replaces a triangular face of a polytope with a tetrahedron, while maintaining the convexity of the embedding (see Fig. 1). A polytope that can be constructed from a tetrahedron and a sequence of stacking operation is called a stacked 3d polytope, or for the scope of this paper simply a *stacked polytope*. The graphs of stacked polytopes are planar 3-trees. Stacked polytopes

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[1] In our terminology polytopes are always considered *convex*.

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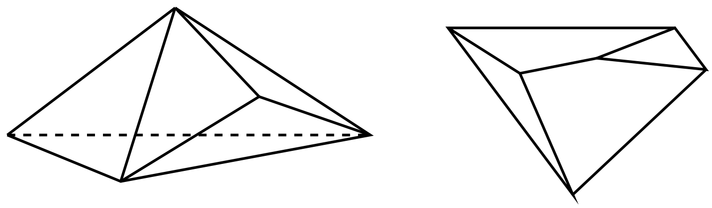

(a) A tetrahedron after one stacking operation.

(b) A tetrahedron after the corresponding truncation.

Fig. 1. The stacking and truncation operations.

can be embedded on a grid that is polynomial in n [6]. This is, however, the only nontrivial polytope class for which such an algorithm is known.

1.1. Our results

In this paper we introduce a duality transform that maintains a polynomial grid size. In other words, we provide a technique that takes a grid embedding of a simplicial polytope with graph G and generates a grid embedding of a polytope whose graph is G^*, the dual graph of G. We call a 3d polytope with graph G^* a *dual polytope.*

We prove the following result:

Theorem *(Theorem 10). Let G be a triangulation with maximal vertex degree Δ_G and let $\mathcal{P} = (u_i)_{1 \le i \le n}$ be a realization of G as a convex polytope with integer coordinates. Then there exists a realization $(\phi_f)_{f \in F(G)}$ of the dual graph G^* as a convex polytope with integer coordinates bounded by*[2]

$$|\phi_f| < O(n \max |u_i|^{3\,\Delta_G + 9}).$$

This, in particular, implies, that if we only consider simplicial polytopes with bounded vertex degrees and with integer coordinates bounded by a polynomial in n, then the dual polytope obtained with our techniques has also integer coordinates bounded by a (different) polynomial in n. Although our bound is not purely polynomial, it is in general an improvement over the standard approaches for constructing dual polytopes; see Sect. 1.2.

For the class of stacked polytopes (although their maximum vertex degree is not bounded) we can also apply our approach to show that all graphs dual to planar 3-trees can be embedded as polytopes on a polynomial size grid. These polytopes are known as truncated polytopes. Truncated 3d polytopes are simple polytopes, which can be generated from a tetrahedron and a sequence of *vertex truncations*. A vertex truncation is the dual operation to stacking (Fig. 1). This means that a degree-3 vertex of the polytope is cut off by adding a new bounding hyperplane that separates this vertex from the remaining vertices of the polytope. We prove the following theorem.

Theorem *(Theorem 4). Any truncated 3d polytope with n vertices can be realized with integer coordinates of size $O(n^{9 \log 6+1})$.*

The proof of Theorem 4 uses the strong available results on planar realizations of graphs of the dual (stacked) polytopes (see [6]).

Such results are not available for general simplicial polytopes, though we make a small step further in Theorem 6. To prove the general Theorem 10 we develop some novel techniques to work with equilibrium stresses directly in $\mathbb{R}^3$.

1.2. Duality

There exist several natural approaches how to construct a dual polytope. To the best of our knowledge, all of them increase the coordinates of the original polytope in general by an exponential factor when scaled to integers.

The most prominent construction is polarity with respect to the sphere. Let P be a polytope that contains the origin. Then $P^* = \{y \in \mathbb{R}^d \colon \langle x, y \rangle \le 1 \text{ for all } x \in P\}$ is a polytope dual to P, called its *polar*. The vertices of P^* are intersection points of planes with integral normal vectors, and hence not necessarily integer points. In order to scale to integrality one has to multiply P^* with the product of all denominators of its vertex coordinates, which may cause an exponential increase of the grid size.

An alternative approach goes via polarity with respect to the paraboloid: Every supporting hyperplane of a polytope facet can be described as the set of points $(x, y, z)^T$, for which the equation $ax + by = z + c$ holds (a, b, c are parameters which

[2] For convenience, throughout the paper we use $|u|$ for the Euclidean norm of the vector u.