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Tolerance limits under normal mixtures: Application to the evaluation of nuclear power plant safety and to the assessment of circular error probable

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ABSTRACT

Upper tolerance limits are derived for (i) a normal mixture distribution, and (ii) for the distribution of the Euclidean norm of a bivariate normal mixture random variable, using asymptotic normality of the MLE and bootstrap calibration. Scenario (i) is used to model the peak cladding temperature (PCT) of nuclear power plants, to assess if the PCT distribution is below a safety threshold of 2200 °F. Scenario (ii) is used to model impact location data on projectiles launched from different locations or systems, and the problem is inference concerning the circular error probable (CEP). Simulation studies show the accuracy of the proposed methodology.

1. Introduction

The present investigation is on the computation of upper tolerance limits under two scenarios of a mixture distribution: (i) a random variable following a univariate mixture distribution where each component is normal, and an upper tolerance limit is to be computed for the distribution, and (ii) a bivariate random variable following a mixture distribution, where each component is bivariate normal, and an upper tolerance limit is to be computed for the distribution of the Euclidean norm of the random variable. The problems have been motivated by two applications, described below.

The scenario in (i) arises in the context of evaluating the safety of nuclear power plants, based on data that are outputs of computer codes used to simulate postulated accidents in nuclear power plants. A major focus in the evaluation of nuclear power plant safety is the *loss of coolant accident* (*LOCA*). The US Nuclear Regulatory Commission (www.nrc.org) defines the LOCA as "Those postulated accidents that result in a loss of reactor coolant at a rate in excess of the capability of the reactor makeup system from breaks in the reactor coolant pressure boundary, up to and including a break equivalent in size to the double-ended rupture of the largest pipe of the reactor coolant system". In the nuclear reactor, nuclear fuel pellets are stacked into metallic tubes, and then sealed. The sealed tubes are called fuel rods, which are used to build up the core of a reactor. The outer protective layer between the nuclear fuel and the coolant is referred to as *cladding*. Clearly, the design, construction, and operation, of nuclear reactors are subject to compliance with regulatory criteria specified for LOCA and other accident scenarios. The criteria are specified in the Code of Federal Regulation 10CFR50.46: *Acceptance Criteria for Emergency Core Cooling Systems* (*ECCS*) *for Light-Water Nuclear Power Reactors*. One criterion to prevent cladding embrittlement is maintaining a peak cladding temperature (PCT) less than 2200 °F. The current practice for verifying if the PCT will remain below the recommended 2200 °F consists of obtaining PCT data from the outputs of computer codes used

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to simulate postulated accidents, and then computing a non-parametric 95% upper confidence limit for the 95th percentile of the PCT distribution; such an upper confidence limit is referred to as an upper tolerance limit having 95% *content* and 95% confidence level. If such an upper tolerance limit is below 2200 °F, it is concluded that the PCT will remain below the recommended 2200 °F with a high probability; see the document by AREVA, Inc. (2010) or the report by Pourgol-Mohammad et al. (2007), where the latter has been prepared for the US Nuclear Regulatory Commission. Other relevant literature where the tolerance limit calculation is explained for PCT data includes the articles by Guba et al. (2003), Nutt and Wallis (2004), Frepoli (2008), and Martin and Nutt (2011). The PCT depends on the breaks in the reactor coolant pressure boundary, and the breaks could be of different sizes. Data on PCT obtained from AREVA, Inc., a company that supplies fuel and provides engineering services for nuclear power plants, show the distribution of the PCT to be bimodal, a minor mode corresponding to the intermediate break sizes, and a major mode corresponding to the large break sizes. Furthermore, there is clear indication that the data is a mixture of two normal distributions. However, the literature on power plant safety evaluation (for example, those cited above) does not use this information on the PCT distribution. It is to be expected that improved tolerance limits can be obtained if we appropriately utilize the normal mixture distributional property. This motivated our investigation of this problem.

The second application that we shall consider is on inference concerning the parameter *circular error probable* (*CEP*), a quantity used to assess the impact accuracy of ballistic projectiles directed at a target. The CEP is defined as the radius of a circle, centered at the target, that is expected to contain 50% of the impact points. A large value of the CEP indicates that impact points could be farther away from the target, compromising accuracy requirements. Thus the accurate estimation of the CEP is of vital interest in the measurement of projectile impact accuracy. In particular, the computation of an upper confidence limit has received considerable attention in the literature. We refer to the recent article by Zhang and An (2012) for a review of the relevant literature. Another application where the CEP is used for evaluating impact accuracy is in the context of bomb drops from an aircraft; see Didonato (2007). For background information on target coverage and weapon impact accuracy, we refer to the reviews by Eckler (1969, 1988), and the report by Eckler and Burr (1972). The CEP has also been recommended for the assessment of firearm performance, especially for comparing various configurations of the equipment (such as different barrels, different loads for cartridges, and different types of pellets for air rifles); see McMillan and McMillan (2008), who make a strong case for using the CEP in order to evaluate rifle performance. A brief review of the literature on the CEP is given in Wollschläger (2014); this article describes the R package `shotGroups`, which can be used for computing the CEP. In addition to applications in the field of ballistics, the CEP has also found ecological applications in the context of assessing the errors in global positioning system (GPS) technology for evaluating wildlife movements; we refer to Moen et al. (1997), D'Eon and Delparte (2005), Lewis et al. (2007) and Frair et al. (2010) for further information.

For estimating the CEP, the data are bivariate, and consist of the deviations of the impact point from the target point, along the horizontal and vertical axes. If $(Y_1, Y_2)'$ is the corresponding bivariate random variable, the CEP is the 50th percentile of its Euclidean norm $\sqrt{Y_1^2 + Y_2^2}$. A 95% upper confidence limit for the CEP is thus an upper tolerance limit for the distribution of $\sqrt{Y_1^2 + Y_2^2}$, with content 0.50 and confidence level 0.95. Typically, the deviations $(Y_1, Y_2)'$ are assumed to follow a bivariate normal distribution. It is easy to see that in the bivariate normal case, the distribution of $Y_1^2 + Y_2^2$ is a linear combination of two independent non-central chisquare random variables, each with one df. However, there are applications where the distribution of $(Y_1, Y_2)'$ is a mixture of bivariate normals; if so, the distribution of $Y_1^2+Y_2^2$ is a mixture where each component in the mixture is a linear combination of two independent non-central chisquare random variables, each having one df. It is this mixture scenario that we have investigated, and we have addressed the problem of computing an upper confidence limit for the CEP parameter, i.e., an upper tolerance limit for the distribution of $Y_1^2 + Y_2^2$. The estimation of the CEP under bivariate normal mixtures is in fact investigated in Spall and Maryak (1992). The authors consider applications where each observation in the sample comes from one of s different bivariate distributions, and an observation comes from the jth distribution with a probability $\lambda_j,\ j = 1, 2, \ldots, s$. As Spall and Maryak (1992) have noted, an example where such a scenario can arise is one where missiles are launched at a specified target from s different locations. The analysis carried out by Spall and Maryak (1992) assumes that the mixing probabilities $\lambda_j,\ j = 1, 2, \ldots, s$, are known. Furthermore, their analysis is Bayesian, and the authors applied their methodology for the assessment of the CEP of ballistic missiles at The Johns Hopkins University, Applied Physics Laboratory. The possibility of using mixture distributions for modeling target impact data is also pointed out in Spall (1997); the author emphasizes the need to have accurate small sample procedures. Our motivation for investigating the CEP problem under a bivariate normal mixture model is based on two applications of interest to the US Army. The first application is on the evaluation of the accuracy of Army mortar systems. The impact data to be used for this purpose consist of shots fired simultaneously at a specified target by several systems during a test. It turns out that the data are distributed as a bivariate normal mixture; this is to be expected since the data are obtained based on several mortar systems. However, in order to assess the accuracy of the overall system, a single CEP value is desired. The second application looks at the miss distances associated with a small arms weapon. This data set consists of bivariate impact data from a single weapon with two different barrels fired during a test. While the miss distances were recorded for each barrel, there is a greater interest in providing an overall CEP for the weapon as opposed to separate CEPs for each barrel. In fact, the calculation of CEP for each barrel is of no interest. Later in the paper, we will be analyzing data based on both of the above applications.

Bivariate normal mixtures also come up in ecological applications where the CEP has to be evaluated in the context of assessing the errors in global positioning system (GPS) technology for evaluating wildlife movements. The distribution of an