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Exact likelihood inference for two exponential populations under joint Type-II censoring

N. Balakrishnan[a,*], Abbas Rasouli[b]

[a]Department of Mathematics and Statistics, McMaster University, Hamilton, Ont, Canada L8S 4K1
[b]Department of Mathematics, Zanjan University, Zanjan 45195, Iran

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Abstract

Comparative lifetime experiments are of paramount importance when the object of a study is to ascertain the relative merits of two competing products in regard to the duration of their service life. In this paper, we discuss exact inference for two exponential populations when Type-II censoring is implemented on the two samples in a combined manner. We obtain the conditional maximum likelihood estimators (MLEs) of the two exponential mean parameters. We then derive the moment generating functions and the exact distributions of these MLEs along with exact confidence intervals and simultaneous confidence regions. Moreover, simultaneous approximate confidence regions based on the asymptotic normality of the MLEs and simultaneous credible confidence regions from a Bayesian viewpoint are also discussed. A comparison of the exact, approximate, bootstrap and Bayesian intervals is also made in terms of coverage probabilities. Finally, an example is presented in order to illustrate all the methods of inference discussed here.

Keywords: Exponential distribution; Joint Type-II censoring; Likelihood inference; Confidence bounds; Simultaneous confidence intervals; Bayes inference; Life-testing; Bootstrap intervals; Coverage probabilities

1. Introduction

The joint censoring scheme is of practical significance in conducting comparative life-tests of products from different units within the same facility. Suppose products are being manufactured by two lines in the same facility. Further, suppose two samples of products of sizes m and n are selected from these two lines and are placed simultaneously on a life-testing experiment. Then, based on cost considerations and also to have a life-test that does not take too long for completion, suppose the experimenter chooses to terminate the life-testing experiment as soon as a certain number of failures occur. In this situation, one may be interested in either point or interval estimation of the mean lifetimes of products manufactured by the two lines and the exact results developed here will facilitate this.

It needs to be mentioned here that this joint censoring scheme has been considered before in the literature. For example, Basu (1968) discussed a generalized Savage statistic while Johnson and Mehrotra (1972) studied locally most powerful rank tests. The problem of testing for the equality of the two distributions, under the assumption of exponentiality, was handled by Bhattacharyya and Mehrotra (1981). Of course, all these developments under this

* Corresponding author. Tel.: +1 905 525 9140; fax: +1 905 522 1676.
E-mail address: bala@mcmaster.ca (N. Balakrishnan).

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sampling scheme, which have been reviewed by Bhattacharyya (1995) in Chapter 7 of Balakrishnan and Basu (1995), have focused on nonparametric and parametric tests of hypotheses. It appears, however, that the exact inference based on the maximum likelihood estimators (MLEs) has not yet been considered and is, therefore, the problem of study in the present work. Let us now assume that specimens of two products under study are placed on a life-test simultaneously, that the successive failure times and the corresponding product types are recorded, and that the experiment is terminated as soon as a specified total number of failures (say, r) occurred.

Suppose $X_1, \ldots, X_m$, the lifetimes of m specimens of product A, are i.i.d. random variables from distribution function $F(x)$ and density function $f(x)$, and $Y_1, \ldots, Y_n$, the lifetimes of n specimens of product B, are i.i.d. random variables from distribution function $G(x)$ and density function $g(x)$. Further, suppose $W_1 \leqslant \cdots \leqslant W_N$ denote the order statistics of the $N = m + n$ random variables $\{X_1, \ldots, X_m; Y_1, \ldots, Y_n\}$. Then, under the joint Type-II censoring scheme, the observable data consist of $(\boldsymbol{Z}, \boldsymbol{W})$, where $\boldsymbol{W} = (W_1, \ldots, W_r)$, with r $(1 \leqslant r < N)$ being a pre-fixed integer, and $\boldsymbol{Z} = (Z_1, \ldots, Z_r)$ with $Z_i = 1$ or 0 according as W_i is from an X- or Y-failure.

In this paper, we consider the case of exponential distribution under such a setting for data and discuss the maximum likelihood estimation of the parameters in Section 2. We then derive the exact conditional moment generating function (cmgf) of the MLEs through which we derive the exact conditional distributions of the MLEs and use them to obtain exact conditional confidence intervals for the parameters. Such exact likelihood inference under exponential distribution has been discussed in a variety of contexts by many authors including Chen and Bhattacharyya (1988), Balakrishnan and Basu (1995), Childs et al. (2003), Chandrasekar et al. (2004), and Balakrishnan et al. (2007). In Section 3, we discuss the construction of approximate confidence intervals based on the asymptotic normality of the MLEs. In Sections 4 and 5, we discuss the Bayesian credible intervals and the bootstrap confidence intervals for the parameters. A Monte Carlo comparison of all these inferential procedures is carried out in Section 6. Finally, an example is presented in Section 7 in order to illustrate all these methods of inference.

2. MLEs, exact distributions and inference

Letting $M_r = \sum_{i=1}^r Z_i$ denote the number of X-failures in $\boldsymbol{W}$ and $N_r = \sum_{i=1}^r (1 - Z_i) = r - M_r$ (i.e., the number of Y-failures in $\boldsymbol{W}$), the likelihood of $(\boldsymbol{Z}, \boldsymbol{W})$ is given by (see, for example, Bhattacharyya, 1995, in Balakrishnan and Basu, 1995, Chapter 7, pp. 93–118)

$$L(\theta_1, \theta_2, \boldsymbol{z}, \boldsymbol{w}) = \frac{m!\, n!}{(m - m_r)!(n - n_r)!} \prod_{i=1}^{r} [\{f(w_i)\}^{z_i} \{g(w_i)\}^{1-z_i}] \{\bar{F}(w_r)\}^{m-m_r} \{\bar{G}(w_r)\}^{n-n_r}, \tag{2.1}$$

where $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$ are the survival functions of the two populations.

In this article, we suppose that the two populations are exponential with

$$\bar{F}(x) = \mathrm{e}^{-x/\theta_1} \quad \text{and} \quad \bar{G}(x) = \mathrm{e}^{-x/\theta_2}, \quad x > 0, \quad \theta_1 > 0, \quad \theta_2 > 0. \tag{2.2}$$

In this case, the likelihood function in (2.1) becomes

$$L(\theta_1, \theta_2, \boldsymbol{z}, \boldsymbol{w}) = \frac{m!\, n!}{(m - m_r)!(n - n_r)!\theta_1^{m_r} \theta_2^{n_r}} \exp\left\{ -\left(\frac{u_1}{\theta_1} + \frac{u_2}{\theta_2} \right) \right\}, \quad 0 < w_1 < \cdots < w_r < \infty, \tag{2.3}$$

where

$$u_1 = \sum_{i=1}^{r} z_i w_i + w_r (m - m_r) = \sum_{i=1}^{m_r} x_{(i)} + w_r (m - m_r),$$
$$u_2 = \sum_{i=1}^{r} (1 - z_i) w_i + w_r (n - n_r) = \sum_{j=1}^{n_r} y_{(j)} + w_r (n - n_r);$$

here, $x_{(\cdot)}$ and $y_{(\cdot)}$ denote the order statistics of the X and Y samples, respectively. The likelihood function $L(\theta_1, \theta_2, \boldsymbol{z}, \boldsymbol{w})$ in (2.3) can be expressed as

$$L(\theta_1, \theta_2, \boldsymbol{z}, \boldsymbol{w}) = L(\theta_1, \boldsymbol{z}, \boldsymbol{w}) \times L(\theta_2, \boldsymbol{z}, \boldsymbol{w}) \tag{2.4}$$