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Short communication

A revisit to the common mean problem: Comparing the maximum likelihood estimator with the Graybill–Deal estimator

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Abstract

For estimating the common mean of two normal populations with unknown and possibly unequal variances the well-known Graybill–Deal estimator (GDE) has been a motivating factor for research over the last five decades. Surprisingly the literature does not have much to show when it comes to the maximum likelihood estimator (MLE) and its properties compared to those of the GDE. The purpose of this note is to shed some light on the structure of the MLE, and compare it with the GDE. While studying the asymptotic variance of the GDE, we provide an upgraded set of bounds for its variance. A massive simulation study has been carried out with very high level of accuracy to compare the variances of the above two estimators results of which are quite interesting.

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1. Introduction

One of the oldest and most interesting problems in statistical inference, which has dogged the researchers over the last five decades, is the estimation of a common mean of two normal populations with unknown and possibly unequal variances.

To be specific, let us assume that we have *i.i.d.* observations $X_{i1}, \ldots, X_{in_i}$ from $N(\mu, \sigma_i^2)$, $i = 1, 2$. Define $\bar{X}_i$ and S_i as

$$\overline{X}_i = \sum_{j=1}^{n_i} X_{ij}/n_i, \quad S_i = \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_i)^2, \tag{1.1}$$

where $\overline{X}_i \sim N(\mu, \sigma_i^2/n_i)$, $S_i \sim \sigma_i^2 \chi^2_{(n_i-1)}$ $(i = 1, 2)$, and these four statistics are mutually independent. Throughout this note it is assumed that $n_i \geqslant 2$ $(i = 1, 2)$ unless mentioned otherwise.

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Note that $(\overline{X}_1, \overline{X}_2, S_1, S_2)$ is minimal sufficient for $(\mu, \sigma_1^2, \sigma_2^2)$ but not complete. As a result, one cannot get the UMVUE (if it exists) using the standard Rao–Blackwell theorem on an unbiased estimator for estimating the common mean μ.

The motivation of this problem (i.e., estimation of μ) comes from a balanced incomplete block design (BIBD) with uncorrelated random block effects. For the ℓth treatment effect (say, τ_ℓ) one has two estimates—namely, the intra-block estimate and the inter-block estimate (say, $\hat{\tau}_\ell$ and $\hat{\tau}_\ell^*$, respectively). Under the usual design assumptions, $\hat{\tau}_\ell$ and $\hat{\tau}_\ell^*$ are independent, have normal distributions with the common mean τ_ℓ but with unknown and possibly unequal variances. The problem thus boils down to derive an efficient estimate of τ_ℓ on the basis of $\hat{\tau}_\ell$, $\hat{\tau}_\ell^*$ and their variance estimates.

Coming back to our original model (1.1), if the population variances $(\sigma_i^2, i = 1, 2)$ are completely known, then the optimal estimator of μ is

$$\hat{\mu} = \sum_{i=1}^{2} (n_i/\sigma_i^2)\overline{X}_i \bigg/ \sum_{i=1}^{2} (n_i/\sigma_i^2) , \tag{1.2}$$

which is the UMVUE, BLUE as well as the MLE. For the case of equal sample sizes, one just needs to know the ratio (σ_1^2/σ_2^2), apart from $\overline{X}_1$ and $\overline{X}_2$, to obtain (1.2).

In our present problem, where σ_i^2, $i=1, 2$, are unknown and possibly unequal, the most appealing unbiased estimator of μ has been the Graybill–Deal estimator (GDE) given as

$$\hat{\mu}_{\mathrm{GD}} = \sum_{i=1}^{2} (n_i/s_i^2)\overline{X}_i \bigg/ \sum_{i=1}^{2} (n_i/s_i^2) , \tag{1.3}$$

where $s_i^2 = S_i/(n_i - 1)$, $i = 1, 2$. Graybill and Deal (1959) obtained conditions on n_1 and n_2 for which $\hat{\mu}_{\mathrm{GD}}$ has a smaller variance than $\overline{X}_i$, $i = 1, 2$.

Even though Graybill and Deal pioneered the research on common mean estimation, it is probably due to Zacks (1966, 1970) that many other researchers paid attention to this interesting problem and its real-life applications. In Zacks' own words—"... In 1963 I was approached by a soil engineer. He wanted to estimate the common mean of two populations and he didn't know anything about the variances. But, apriori from his theory he said that the means should be same, and here are the two samples from two different soils. So I thought about this problem a little bit and I started to investigate. I realized that there is room for innovation ..." (see Kempthorne et al., 1991).

For other applications of the common mean problem, especially in clinical trials, see Kelleher (1996).

Most of the research so far on the estimation of a common mean has taken place on three fonts: (i) comparing the GDE with the individual sample means (i.e., $\overline{X}_i$'s); (ii) studying the optimality of GDE and its natural generalizations in suitable classes of estimators; and (iii) studying the performance of Bayes and preliminary test-based estimators with that of the GDE. For a good review of the literature on this problem and other generalizations one can see Kubokawa (1987, 1991) and other references therein. Among some interesting results pertaining to the GDE, Sinha (1985) obtained an unbiased estimate of the variance of the GDE in the form of an infinite series which can be truncated suitably to get an approximate unbiased estimate up to any desired order. This result is helpful because the studentized version $[\{(\hat{\mu}_{\mathrm{GD}} - \mu)/V\hat{a}r(\hat{\mu}_{\mathrm{GD}})\}^{1/2}]$, which follows $N(0, 1)$ asymptotically, can be used for testing as well as for interval estimation of μ.

Quite surprisingly there has not been any discussion about the MLE and its performance relative to the other estimators, especially the GDE. It should be pointed out that the GDE (in (1.3)) is not the MLE, contrary to the statement made by Kelleher (1996) or Sinha (1979).

The purpose of this note is to focus on the MLE and its properties which have long been neglected in the literature. Even with the availability of affordable and efficient computing resources no comparison has been made so far to see how the GDE performs relative to the MLE. An important component of our study has been to see how the variances of the GDE and the MLE depend on the parameters as well as the sample sizes. The numerical results that are reported in the literature did not take this aspect seriously. As a result, the reported numerical studies have been haphazard, or incomplete at best.

In Section 2, we study the structure of the MLE and provide a useful representation. Also we find its bias (exact) and variance expressions. Further, we upgrade the existing results to obtain tighter bounds for the variance of the GDE. In Section 3, we report the results of our extensive numerical study comparing the variances of the GDE and the MLE. A