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Families of polytopal digraphs that do not satisfy the shelling property

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ABSTRACT

A polytopal digraph $G(P)$ is an orientation of the skeleton of a convex polytope P. The possible non-degenerate pivot operations of the simplex method in solving a linear program over P can be represented as a special polytopal digraph known as an LP digraph. Presently there is no general characterization of which polytopal digraphs are LP digraphs, although four necessary properties are known: acyclicity, unique sink orientation (USO), the Holt–Klee property and the shelling property. The shelling property was introduced by Avis and Moriyama (2009), where two examples are given in $d = 4$ dimensions of polytopal digraphs satisfying the first three properties but not the shelling property. The smaller of these examples has $n = 7$ vertices. Avis, Miyata and Moriyama (2009) constructed for each $d \geqslant 4$ and $n \geqslant d + 2$, a d-polytope P with n vertices which has a polytopal digraph which is an acyclic USO that satisfies the Holt–Klee property, but does not satisfy the shelling property. The construction was based on a minimal such example, which has $d = 4$ and $n = 6$. In this paper we explore the shelling condition further. First we give an apparently stronger definition of the shelling property, which we then prove is equivalent to the original definition. Using this stronger condition we are able to give a more general construction of such families. In particular, we show that given any 4-dimensional polytope P with n_0 vertices whose unique sink is simple, we can extend P for any $d \geqslant 4$ and $n \geqslant n_0 + d - 4$ to a d-polytope with these properties that has n vertices. Finally we investigate the strength of the shelling condition for d-crosspolytopes, for which Develin (2004) has given a complete characterization of LP orientations.

1. Introduction

Let P be a d-dimensional convex polytope (d-polytope) in $\Re^d$. We assume that the reader is familiar with polytopes, a standard reference being [13]. The vertices and extremal edges of P form an (abstract) undirected graph called the skeleton of P. A *polytopal digraph* $G(P)$ is formed by orienting each edge of the skeleton of P in some manner. In the paper, when we refer to a polytopal digraph $G(P)$ we shall mean the pair of both the digraph and the polytope P itself, not just the abstract digraph.

We can distinguish four properties that the digraph $G(P)$ may have, each of which has been well studied:

- *Acyclicity*: $G(P)$ has no directed cycles.

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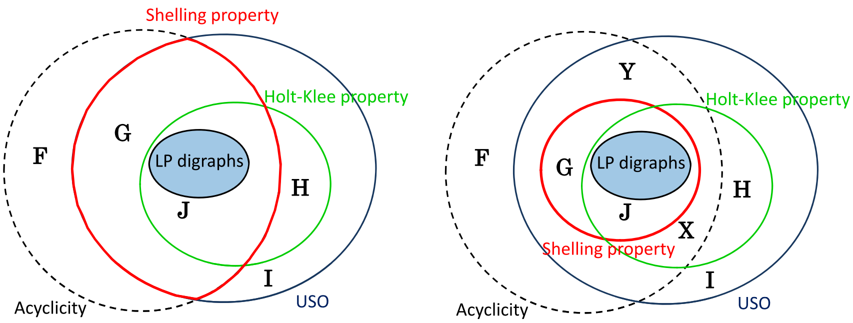

Fig. 1. The relationships among the four properties when P is simple [left] and general [right].

- *Unique sink orientation* (*USO*) (Szabó and Welzl [12]): Each subdigraph of $G(P)$ induced by a non-empty face of P has a unique source and a unique sink.
- *Holt–Klee property* (Holt and Klee [10]): $G(P)$ has a unique sink orientation, and for every $k(\geqslant 0)$-dimensional face (k-face) H of P there are k vertex disjoint paths from the unique source to the unique sink of H in the subdigraph $G(P, H)$ of $G(P)$ induced by H.
- *LP digraph*: There is a linear function f and a polytope P' combinatorially equivalent to P such that for each pair of vertices u and v of P' that form a directed edge (u, v) in $G(P')$, we have $f(u) < f(v)$. (LP digraphs are called polytopal digraphs in Mihalisin and Klee [11].)

Interest in polytopal digraphs stems from the fact that the simplex method with a given pivot rule can be viewed as an algorithm for finding a path to the sink in a polytopal digraph which is an LP digraph. Research continues on pivot rules for the simplex method since they leave open the possibility of finding a strongly polynomial time algorithm for linear programming. An understanding of which polytopal digraphs are LP digraphs is therefore of interest. The other three properties are necessary properties for $G(P)$ to be an LP digraph. We note here that Williamson Hoke [9] has defined a property called *complete unimodality* which is equivalent to a combination of acyclicity and unique sink orientation. LP digraphs are completely characterized when $d = 2, 3$. Their necessary and sufficient properties in dimension $d = 2$ are a combination of acyclicity and unique sink orientation, i.e., complete unimodality, and those in $d = 3$ are a combination of acyclicity, unique sink orientation and the Holt–Klee property [11]. On the other hand, no such characterization is known yet for higher dimensions.

Another necessary property for $G(P)$ to be an LP digraph is based on *shelling*, which is one of the fundamental tools of polytope theory. A formal definition of shelling is given in Section 2. Let $G(P)$ be a polytopal digraph for which the polytope P has n vertices labelled $v_1, v_2, \ldots, v_n$. A permutation r of the vertices is a *topological sort* of $G(P)$ if, whenever (v_i, v_j) is a directed edge of $G(P)$, v_i precedes v_j in the permutation r. Let $L(P)$ be the face lattice of P. A polytope P^* whose face lattice is $L(P)$ “turned upside-down” is called a *combinatorially polar* polytope of P. Combinatorial polarity interchanges vertices of P with facets of P^*. We denote by r^* the facet ordering of P^* corresponding to the vertex ordering of P given by r.

- *Shelling property* (Avis and Moriyama [3]): There exists a topological sort r of $G(P)$ such that the facets of P^* ordered by r^* are a shelling of P^*.

Results relating these properties of polytopal digraphs have been obtained by various authors. Fig. 1 summarizes the relationship among the 4 properties, where the regions F, G, H, I, J, X, Y are non-empty. For further information, see [3].

When we restrict ourselves to the case $d \leqslant 3$, the region X is empty since the shelling property is characterized by acyclic USOs for $d \leqslant 3$ [3]. The region X is also empty for simple polytopes because the shelling property is equivalent to the acyclic USO property for simple polytopes [3,9]. On the other hand, two examples of non-simple polytopal digraphs in the region X had been known for $d = 4$: Develin's example [6] is a polytopal digraph on the skeleton of a 4-dimensional crosspolytope with eight vertices, and the example proposed by Avis and Moriyama [3] is a polytopal digraph a 4-polytope with seven vertices.

The existence of the non-empty region X shows the importance of the shelling property, namely that there exist polytopal digraphs satisfying the three existing necessary properties for LP digraphs, but not the shelling property. This motivates the following definition.

Definition 1. A polytopal graph $G(P)$ is called an *X-type graph* if it is an acyclic USO satisfying the Holt–Klee property, but not the shelling property.

In our paper in the proceedings of 6th Japanese–Hungarian Symposium on Discrete Mathematics and Its Applications [2], we proved the following result.