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Simulation of Lévy-driven Ornstein–Uhlenbeck processes with given marginal distribution

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Abstract

A simulation procedure for obtaining discretely observed values of Ornstein–Uhlenbeck processes with given (self-decomposable) marginal distribution is provided. The method proposed, based on inversion of the characteristic function, completely circumvents the problems encountered when trying to reproduce small jumps of Lévy processes. Error bounds for the proposed procedure are provided and its performance is numerically assessed.

1. Introduction

The modeling via the use of Lévy processes has received considerable attention in recent literature in an attempt to accommodate features such as jumps, semi-heavy tails and asymmetry which are often present in real phenomena and are a point of remarkable interest in fields of application such as finance and economics.

Among recent contributions we find a completely new class of models, termed non-Gaussian Ornstein–Uhlenbeck (OU) models by which, stochastic processes with given correlation structure and (possibly non-Gaussian) marginal distribution are constructed by means of self-decomposability; see Barndorff-Nielsen (1998, 2001), Barndorff-Nielsen et al. (1998), Barndorff-Nielsen and Shephard (2001, 2003), Schoutens (2003) and Barndorff-Nielsen and Leonenko (2005).

The availability of simulation techniques of easy implementation is important for analysis, validation and estimation purposes. For example, since direct likelihood analysis is often impracticable for these models, simulation based techniques such as Bayesian and method-of-moments approaches can be a viable route to estimation.

In this article we develop and assess practical schemes to simulate non-Gaussian OU processes with given self-decomposable marginal distribution and given auto-correlation structure. We use a simulation scheme based on numerical inversion of the characteristic function (ch.f.) which turns out to be considerably simple to implement. The convenience in using the ch.f. is especially apparent for OU processes: for example, the Normal Inverse Gaussian OU process is composed by the sum of three independent Lévy processes; existing methods require to simulate each

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process separately and, given explicit expressions of the relevant quantities are not always available, one needs to resort to approximations or methods which require quite a bit of analytical work to implement. As we will see, the use of the ch.f. will completely circumvent these problems; moreover explicit expressions of the ch.f. are available for several important models.

If an analytic expression for the inversion formula for the ch.f. would be available, such as in the Gaussian case, the proposed simulation scheme would provide exact simulation of OU processes; however, this is not the case for many non-Gaussian models of interest and one needs to resort to numerical inversion. We provide bounds and rules to control the numerical error of the inversion. Indeed, given enough computing time, any desired level of accuracy may be attained. Numerical results of inversions can be stored and used as many times as needed to quickly obtain simulated values of OU processes.

The remainder of the article is organized as follows. Section 2 sets forth some notation and recalls the properties of OU processes which we will need in order to develop the simulation scheme. Section 3 sets out the simulation scheme for OU processes with self-decomposable distribution. Section 4 presents some examples to assess the accuracy of the proposed procedure.

2. Background

The present section reviews some known results; for further details and generalizations the reader is referred to Wolfe (1982), Barndorff-Nielsen et al. (1998), Barndorff-Nielsen (1998, 2001) and Sato (1999).

Recall that a random variable X is self-decomposable if, for any $c \in (0, 1)$, there exists a ch.f. $\phi_c(\zeta)$ such that the ch.f. of X, $\phi(\zeta)$, can be decomposed as

$$\phi(\zeta) = \phi(c\zeta)\phi_c(\zeta). \tag{1}$$

An OU process $\{X(t), t \geq 0\}$, given $\lambda > 0$ and a homogeneous Lévy process $\grave{Z}(t)$ for which $\mathrm{E}[\log(1 + |\grave{Z}(1)|)] < \infty$, satisfies the differential equation (see, for example, Barndorff-Nielsen (1998))

$$\mathrm{d}X(t) = -\lambda X(t)\mathrm{d}t + \mathrm{d}\grave{Z}(t); \tag{2}$$

$\grave{Z}(t)$ is commonly referred to as the background driving Lévy process (BDLP). Eq. (2) has a strong solution

$$X(t) = \mathrm{e}^{-\lambda t}X(0) + \int_0^t \mathrm{e}^{-\lambda(t-s)}\mathrm{d}\grave{Z}(s). \tag{3}$$

Up to indistinguishability, this solution is unique (Sato, 1999, Section 17). We assume that the Levy process $\{\grave{Z} = \grave{Z}(t), t \geq 0\}$ has right-continuous sample paths, with existing left-hand limits. Furthermore, since $X(t)$ is given as a stochastic integral with respect to a cádlág semi-martingale, the OU process $\{X(t), t \geq 0\}$ can be assumed to be cádlág itself. If ν denotes the Lévy measure of $\grave{Z}$, under the condition $\int_{|x|\leq 1} |x|\nu(\mathrm{d}x) < \infty$, the stochastic integral in (3) can be interpreted as a pathwise Lebesgue–Stieltjes integral, since the paths of $\grave{Z}$ are almost surely of finite variation on each interval $(0, t]$, $t \geq 0$ (Sato, 1999, Theorem 21.9). This holds for all examples discussed here; for other interesting interpretations and discussions about the stochastic integral in (3), see Anh et al. (2002), p. 733.

If $X(t)$ is to be stationary, the ch.f. of its marginal distribution must have the form $\phi(\zeta) = \phi(\mathrm{e}^{-\lambda t}\zeta)\phi_t(\zeta)$ for all $t \geq 0$, where $\phi_t(\zeta)$ denotes the ch.f. of the second term on the right-hand side of (3). Hence the marginal law of $X(t)$ must be self-decomposable. It turns out that there is a precise relation between $\phi(\zeta)$ and $\phi_t(\zeta)$ that is, for the process $X(t)$ to have marginal distribution with ch.f. $\phi(\zeta)$ for all $t \geq 0$ then (Barndorff-Nielsen et al., 1998, Lemma 3.1)

$$\phi_t(\zeta) = \exp\left\{\int_{\zeta \mathrm{e}^{-\lambda t}}^{\zeta} \kappa'(w)\mathrm{d}w\right\} = \exp\{\kappa(\zeta) - \kappa(\zeta \mathrm{e}^{-\lambda t})\}, \tag{4}$$

where $\kappa(\zeta) = \log \phi(\zeta)$ is the cumulant function of X. In other words, for each $t \geq 0$, we are able to define the ch.f. of the error term, given a required marginal (self-decomposable) distribution of the process $X(t)$. This will be the key point for our simulation method.

For estimation purposes one is forced to observe (and simulate) an OU process at time instants t_j, $j = 1, \ldots, n$. For convenience we will denote $X(t_j) = X_j$ and consider equidistant observations $t_j - t_{j-1} = \tau$, $j = 1, \ldots, n$. This, in practice, does not change the salient features of the process which are important in estimation. In fact, as