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MCMC-based local parametric sensitivity estimations

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Abstract

Bayesian inferences for complex models need to be made by approximation techniques, mainly by Markov chain Monte Carlo (MCMC) methods. For these models, sensitivity analysis is a difficult task. A novel computationally low-cost approach to estimate local parametric sensitivities in Bayesian models is proposed. This method allows to estimate the sensitivity measures and their errors with the same random sample that has been generated to estimate the quantity of interest. Conditions to allow a derivative-integral interchange in the operator of interest are required. Two illustrative examples have been considered to show how sensitivity computations with respect to the prior distribution and the loss function are easily obtained in practice.

Keywords: Bayesian decision theory; Bayesian inference; MCMC; Simulation

1. Introduction

Bayesian statistics has become more popular, thanks to the appearance of Markov chain Monte Carlo (MCMC) methods (see Brooks, 1998 for a review and Gilks et al., 1998 for a monograph). The application of these simulation techniques allows to obtain a numerical solution of problems based on really complex models. Sometimes, MCMC methods are the only computationally efficient alternative.

When performing a Bayesian analysis, inferences depend on some input models as the prior distribution, the likelihood or the loss function. Besides the model solution, some description of its sensitivity to the specification of these inputs is necessary. Sensitivity of inferences to the choice of the prior distribution has been widely investigated (see, for example, Berger, 1994). Sivaganesan (1993) and Dey et al. (1996) studied sensitivity with respect to the prior and the likelihood. Martín et al. (1998) considered the loss function and Martín et al. (2003) investigated joint sensitivity with respect to utility and prior distribution. Two relevant monographs on robust Bayesian analysis are provided by Berger et al. (1996) and Ríos Insua and Ruggeri (2000).

Sensitivity analysis can be studied from two viewpoints: local and global. Local sensitivity considers the behavior of posterior quantities of interest under infinitesimal perturbations from a specified input (prior or loss in this paper). On the other hand, global sensitivity quantifies the range of a posterior quantity when the prior (loss) varies in a class. See Sivaganesan (2000) for a comparative review on the local and global approaches to Bayesian robustness.

Sensitivity analyses are demanded by several authors to be applied in complex models that need to be solved by MCMC methods (see, for example, Ríos Insua and Ruggeri, 2000). Some authors, like Richarson and Green (1997), Hall

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et al. (2003), Halekoh and Vach (2003) study parametric sensitivity by solving the model for some values of the prior parameters. They, basically, re-run the Markov chain for different parameters of the prior distributions and estimate the quantities of interest under those parameter specifications. This is computationally costly and, generally, not enough. Therefore, it would be convenient to develop a general method that can be applied to estimate local sensitivities. This is the issue addressed in the paper.

The outline of the paper is as follows. In Section 2, the problem is described and some relevant results on local sensitivity analysis are summarized. Section 3 focuses on the parametric sensitivity case. A computationally low-cost method to estimate local parametric sensitivities in models solved by MCMC methods is proposed. Section 4 presents two illustrative examples which show how the proposed method is easily applied in practice. A discussion is presented in Section 5. Finally, an appendix containing the proofs is included.

2. Local sensitivity estimations

Suppose the interest is focused on the estimation of a quantity that can be expressed as the integral of a real valued function *f* over a multiple dimension domain with respect to a density *p*, i.e.

$$\int_{\Theta} f(\theta)p(\theta)\,\mathrm{d}\theta. \tag{1}$$

When *p* is the posterior distribution for θ, i.e., $p(\theta|x)$, the integral (1) becomes the posterior expectation of $f(\theta)$. In this case, the posterior mean is recovered when *f* is the identity function.

In Bayesian decision theory and inference, the elicitation of the prior distribution $\pi(\theta)$ (or $f(\theta)$ when it denotes the loss function) is far from simple. A class of prior distributions (or loss functions) is usually considered in robustness analysis. Then, expression (1) is denoted by

$$\mathscr{I}(\pi, f) = \frac{\int_{\Theta} f(\theta)l(\theta|x)\pi(\theta)\,\mathrm{d}\theta}{\int_{\Theta} l(\theta|x)\pi(\theta)\,\mathrm{d}\theta}, \quad \pi \in \Gamma, \quad f \in \mathscr{F}, \tag{2}$$

where Γ denotes a class of prior distributions and $\mathscr{F}$ represents a class of functions. Note that the likelihood $l(\theta|x)$ is considered fixed and both classes must be chosen to make (2) integrable.

Suppose that sampling directly from $p(\theta|x)$ is so complex that the use of MCMC methods is necessary. Note that this is the most usual case for real problems in Bayesian inference. Let $\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)}$ be a sample generated from $p(\theta|x)$ by MCMC methods, where $\pi(\cdot) \in \Gamma$ and $f \in \mathscr{F}$ are fixed. Then, an estimate of $\mathscr{I}(\pi, f)$ is given by

$$\widehat{\mathscr{I}(\pi, f)} = \frac{1}{n}\sum_{i=1}^{n} f\left(\theta^{(i)}\right).$$

Now, the interest is focused on evaluating the impact on $\mathscr{I}(\pi, f)$ when π (or *f*) varies in a neighborhood, i.e., a local sensitivity analysis is required. The choice of a sensitivity analysis method depends on a great extent on (a) the sensitivity measures employed, (b) the accuracy in the estimates of the sensitivity measures, and (c) the computational cost involved. The rest of this section is devoted to measures used in local functional sensitivity, whereas next section deals with the particular case of local parametric sensitivity.

The use of derivatives is widely studied in sensitivity literature. Diaconis and Freedman (1986) proposed to use Fréchet derivatives with respect to the prior distribution in local sensitivity analysis. They proposed to use the norm of the derivative as a sensitivity measure. Cuevas and Sanz (1988) extended the results and provided a mathematical framework. Other local approaches based on ε-contaminated classes or Gateaux derivatives are presented in Srinivasan and Truszczynska (1990), Ruggeri and Wasserman (1995), Dey et al. (1996) and Sivaganesan (2000), among others.

Some related definitions and results are summarized below. The notation is the following: the set of parameters endowed with a σ-algebra $\mathscr{B}$ is denoted as Θ, the likelihood is $l(\theta|x)$, where x is the data vector, the class of prior distributions is Γ, the objective function belonging to a class $\mathscr{F}$ is represented by $f(\theta)$, and, finally, the set of signed measures with total mass zero over $(\Theta, \mathscr{B})$ is denoted by $\mathscr{M}$.