Available online at www.sciencedirect.com
ScienceDirect
Computational Statistics & Data Analysis 51 (2006) 47 – 69
COMPUTATIONAL STATISTICS & DATA ANALYSIS
www.elsevier.com/locate/csda

Possibility theory and statistical reasoning

Didier Dubois*

Institut de Recherche en Informatique de Toulouse, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 4, France

Available online 11 May 2006

Abstract

Numerical possibility distributions can encode special convex families of probability measures. The connection between possibility theory and probability theory is potentially fruitful in the scope of statistical reasoning when uncertainty due to variability of observations should be distinguished from uncertainty due to incomplete information. This paper proposes an overview of numerical possibility theory. Its aim is to show that some notions in statistics are naturally interpreted in the language of this theory. First, probabilistic inequalites (like Chebychev's) offer a natural setting for devising possibility distributions from poor probabilistic information. Moreover, likelihood functions obey the laws of possibility theory when no prior probability is available. Possibility distributions also generalize the notion of confidence or prediction intervals, shedding some light on the role of the mode of asymmetric probability densities in the derivation of maximally informative interval substitutes of probabilistic information. Finally, the simulation of fuzzy sets comes down to selecting a probabilistic representation of a possibility distribution, which coincides with the Shapley value of the corresponding consonant capacity. This selection process is in agreement with Laplace indifference principle and is closely connected with the mean interval of a fuzzy interval. It sheds light on the "defuzzification" process in fuzzy set theory and provides a natural definition of a subjective possibility distribution that sticks to the Bayesian framework of exchangeable bets. Potential applications to risk assessment are pointed out.

Keywords: Possibility theory; Imprecise probability; Confidence intervals; Uncertainty propagation

1. Introduction

There is a continuing debate in the philosophy of probability between subjectivist and objectivist views of uncertainty. Objectivists identify probabilities with limit frequencies and consider subjective belief as scientifically irrelevant. Conversely subjectivists consider that probability is tailored to the measurement of belief, and that subjective knowledge should be used in statistical inference. Both schools anyway agree on the fact that the only reasonable mathematical tool for uncertainty modelling is the probability measure. Yet, the idea, put forward by Bayesian subjectivists, that it is always possible to come up with a precise probability model, whatever the problem at hand, looks debatable. This claim can be challenged due to the simple fact that there are at least two kinds of uncertain quantities: those which are subject to intrinsic variability (the height of adults in a country), and those which are totally deterministic but anyway ill-known, either because they pertain to the future (the date of the death of a living person), or just because of a lack of knowledge (I may not know in this moment the precise age of the President). It is clear that the latter cause of uncertainty

* Tel.: +33 61 556 331; fax: +33 61 556 239.
E-mail address: Dubois@irit.fr.

0167-9473/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.csda.2006.04.015

is not "objective", because when lack of knowledge is at stake, it is always somebody's knowledge, which may differ from somebody else's knowledge. However, it is not clear that incomplete knowledge should be modelled by the same tool as variability (a unique probability distribution) (see Hoffman and Hammonds, 1994; Dubois et al., 1996; Ferson and Ginzburg, 1996). One may argue with several prominent other scientists like Dempster (1967) or Walley (1991) that the lack of knowledge is precisely reflected by the situation where the probability of events is ill-known, except maybe for a lower and an upper bound. Moreover one may also have incomplete knowledge about the variability of a non-deterministic quantity if the observations made were poor, or if only expert knowledge is available. This point of view may to some extent reconcile subjectivists and objectivists: it agrees with subjectivists that human knowledge matters in uncertainty judgements, but it concedes to objectivists that such knowledge is generally not rich enough to allow for a full-fledged probabilistic modelling.

Possibility theory is one of the current uncertainty theories devoted to the handling of incomplete information, more precisely it is the simplest one, mathematically. To a large extent, it is similar to probability theory because it is based on set functions. It differs from the latter by the use of a pair of dual set functions called possibility and necessity measures (Dubois and Prade, 1980) instead of only one. Besides, it is not additive and makes sense on ordinal structures. The name "Theory of Possibility" was coined by Zadeh (1978). In Zadeh's view, possibility distributions were meant to provide a graded semantics to natural language statements. However, possibility and necessity measures can also be the basis of a full-fledged representation of partial belief that parallels probability. It can be seen either as a coarse, non-numerical version of probability theory, or a framework for reasoning with extreme probabilities (Spohn, 1988), or yet a simple approach to reasoning with imprecise probabilities (Dubois and Prade, 1992). The theory of large deviations in probability theory also handles set functions that look like possibility measures (Nguyen and Bouchon-Meunier, 2003). Formally, possibility theory refers to the study of *maxitive and minitive* set functions, respectively, called *possibility and necessity measures* such that the possibility degree of a disjunction of events is the maximum of the possibility degrees of events in the disjunction, and the necessity degree of a conjunction of events is the minimum of the necessity degrees of events in the conjunction. There are several branches of possibility theory, some being qualitative, others being quantitative, all satisfying the maxitivity and minitivity properties. But the variants of possibility theory differ for the conditioning operation. This survey focuses on numerical possibility theory. In this form, it looks of interest in the scope of coping with imperfect statistical information, especially non-Bayesian statistics relying on likelihood functions (Edwards, 1972), and confidence or prediction intervals. Numerical possibility theory provides a simple representation of special convex sets of probability functions in the sense of Walley (1991), also a special case of Dempster's upper and lower probabilities (Dempster, 1967), and belief functions of Shafer (1976). Despite its radical simplicity, this framework is general enough to model various kinds of information items: numbers, intervals, consonant (nested) random sets, as well as linguistic information, and uncertain formulae in logical settings (Dubois et al., 2000b).

Just like probabilities being interpreted in different ways (e.g., frequentist view vs. subjective view), possibility theory can support various interpretations. Hacking (1975) pointed out that possibility can be understood either as an objective notion (referring to properties of the physical world) or as an epistemic one (referring to the state of knowledge of an agent). Basically there are four ideas each of which can be conveyed by the word "possibility". First is the idea of feasibility, such as ease of achievement, also referring to the solution to a problem, satisfying some constraints. At the linguistic level, this meaning is at work in expressions such as "it is possible to solve this problem". Another notion of possibility is that of plausibility, referring to the propensity of events to occur. At the grammatical level, this semantics is expressed by means of sentences such as "it is possible *that* the train arrives on time". Yet another view of possibility is logical and it refers to consistency with available information. Namely, stating that a proposition is possible means that it does not contradict this information. It is an all-or-nothing version of plausibility. The last semantics of possibility is deontic, whereby possible means allowed, permitted by the law. In this paper, we focus on the epistemic view of possibility, which also relies on the idea of logical consistency. In this view, possibility measures refer to the idea of plausibility, while the dual necessity functions attempt to quantify the idea of certainty. Plausibility is dually related to certainty, in the sense that the certainty of an event reflects a lack of plausibility of its opposite. This is a striking difference with probability which is self-dual. The expression *It is not probable that* "*not A*" is equivalent to saying *It is probable that A*, while the statement *It is not possible that* "*not A*" is not equivalent to saying *It is possible that A*. It has a stronger meaning, namely: *It is necessary that A*. Conversely, asserting that *it is possible that A* does not entail anything about the possibility nor the impossibility of "not *A*". Hence we need a dual pair of possibility and necessity functions.