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On the Marshall–Olkin transformation as a skewing mechanism

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ABSTRACT

The use of the Marshall–Olkin transformation as a skewing mechanism is investigated. The distributions obtained when this transformation is applied to several classes of symmetric and unimodal distributions are analysed. It is shown that most of the resulting distributions are not flexible enough to model data presenting high or moderate skewness. The only case encountered where the Marshall–Olkin transformation can be considered a useful skewing mechanism is when applied to Student-*t* distributions with Cauchy or even heavier tails.

1. Introduction

The need for modelling data presenting departures from symmetry has fostered the development of more flexible classes of distributions. A popular approach is to modify a symmetric distribution by introducing a parameter that controls skewness (Azzalini, 1985; Fernández and Steel, 1998; Jones, 2004; Ferreira and Steel, 2006).

In the context of reliability and survival analysis, Marshall and Olkin (1997) proposed a transformation of a distribution $F(x;\theta)$ that introduces a new parameter $\gamma > 0$. This transformation is defined through the cumulative distribution function (cdf)

$$G(x;\theta,\gamma) = \frac{F(x;\theta)}{F(x;\theta)+\gamma(1-F(x;\theta))}, \tag{1}$$

and assuming continuity of F throughout, the corresponding probability density function (pdf) is given by

$$g(x;\theta,\gamma) = \frac{\gamma f(x;\theta)}{[F(x;\theta)+\gamma(1-F(x;\theta))]^2}. \tag{2}$$

The interpretation of the parameter γ is given in Marshall and Olkin (1997) in terms of the behaviour of the ratio of hazard rates of F and G. This ratio is increasing in x for $\gamma \geq 1$ and decreasing in x for $0 < \gamma \leq 1$. This transformation is then proposed for the Exponential and Weibull distribution in Marshall and Olkin (1997) in order to generate more flexible models for lifetime data. Clearly, for $\gamma = 1$, G and F coincide.

Using the fact that the distribution in (1) describes a wider class than the original distribution F, García et al. (2010) define a generalised normal distribution (GN) by applying this transformation to a normal distribution F. They investigate the role of γ as a skewness parameter using the standardised third central moment $\text{EM} = \mu_3/\mu_2^{3/2}$ as a skewness measure (Edgeworth, 1904). In a similar search for families of skewed distributions, George and George (forthcoming) apply the Marshall–Olkin

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transformation to the characteristic function of an Esscher transformed Laplace distribution (which, interestingly, leads to a very simple two-piece distribution with inverse scale factors, used later to generate data in Section 3.2). Maiti and Dey (2012) propose exactly the same distribution as the GN of García et al. (2010) and call it the tilted normal distribution. However, they focus mostly on its use for modelling survival data and less on the skewness properties.

We will focus here on the use of the Marshall–Olkin transformation in (1) as a mechanism for inducing skewness in symmetric and unimodal distributions F which are defined over the entire real line. It is immediate from (2) that $g(x;\theta,\gamma) = g(-x;\theta,1/\gamma)$, which means that usual measures of skewness will change sign by inverting γ and that superficially suggests γ plays the part of a skewness parameter. Perhaps the most obvious choice for F is the normal, as explored by García et al. (2010) and Maiti and Dey (2012), and we will first investigate the wider class of Student-t distributions.

In Section 2 we study the tail behaviour induced by the Marshall–Olkin transformation and in the next section we define a generalised t distribution based on the transformation in (1). We explore the role of the parameter γ in the generalised t and the generalised normal distributions using different measures of skewness and we show that the standardised third central moment can lead to counterintuitive conclusions about the shape of the density. In fact, if we use a different measure of skewness based on the relative mass both sides of the mode, it becomes clear that the Marshall–Olkin transformation applied to normal and Student-t distributions with tails that are not extremely fat is unable to accommodate even moderate amounts of skewness. Section 3.2 illustrates this with some simulated data. Section 4 examines the use of the Marshall–Olkin transformation on other classes of distributions and Section 5 provides some intuitive explanation of the observed behaviour. Finally, we conclude that the Marshall–Olkin transformation cannot generally be used as a skewing mechanism for unimodal symmetric distributions, and we find only one exception: the Student-t distribution with Cauchy or even heavier tails.

2. Tail behaviour

Marshall and Olkin (1997) proved existence of moments of (1) for the cases when F is Exponential or Weibull. The next Theorem shows that this transformation preserves moment existence for general F.

Theorem 1. *The moments of (1) exist for exactly the same order as in the original distribution F.*

Proof. Note that if $\gamma < 1$, then

$$\gamma < \frac{\gamma}{[F(x;\theta)+\gamma(1-F(x;\theta))]^2} < \frac{1}{\gamma}.$$

If $\gamma > 1$, then

$$\frac{1}{\gamma} < \frac{\gamma}{[F(x;\theta)+\gamma(1-F(x;\theta))]^2} < \gamma.$$

Therefore

$$g(x;\theta,\gamma) = K(x,\theta,\gamma)f(x;\theta),$$

where $K(x,\theta,\gamma)$ takes values in between $\min\{\gamma, 1/\gamma\}$ and $\max\{\gamma, 1/\gamma\}$. The result follows. □

Theorem 1 shows that transformation (1) produces a distribution with exactly the same tail behaviour as the original.

3. Generalised *t*

We now define a generalised t (Gt) distribution by applying the Marshall–Olkin transformation to the Student-t distribution.

Definition 2. A random variable X is distributed according to the generalised t distribution if its cdf and pdf are given by

$$\mathrm{Gt}(x;\mu,\sigma,\nu,\gamma) = \frac{F(x;\mu,\sigma,\nu)}{F(x;\mu,\sigma,\nu)+\gamma(1-F(x;\mu,\sigma,\nu))}, \quad (3)$$

$$\mathrm{gt}(x;\mu,\sigma,\nu,\gamma) = \frac{\gamma f(x;\mu,\sigma,\nu)}{[F(x;\mu,\sigma,\nu)+\gamma(1-F(x;\mu,\sigma,\nu))]^2}, \quad (4)$$

where F and f are the cdf and pdf of a Student-t distribution with location μ, scale σ and ν degrees of freedom.

Fig. 1 shows some examples of density (4) for different choices of the parameters. Of course, panel (a) is just the Student-t, whereas panel (b) corresponds to $\gamma = 0.5$ and (c) is for $\gamma = 2$. Visually, two things are worth noting about Fig. 1: the densities generated do not seem highly skewed (even though γ is rather far from one), especially for larger values of ν, and the amount of skewness seems to depend on the value of ν. This would suggest that ν and γ cannot straightforwardly be assigned roles as tail and skewness parameters, respectively.

Just as in the symmetric case, the generalised normal distribution (GN) (García et al., 2010) is a limiting case of the Gt distribution, since $\lim_{\nu\to\infty}\mathrm{Gt}(x;\mu,\sigma,\nu,\gamma) = \mathrm{GN}(x;\mu,\sigma,\gamma)$.