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Approximate confidence interval for standard deviation of nonnormal distributions[☆]

Douglas G. Bonett*

Department of Statistics, Iowa State University, Snedechor Hall, Ames, IA 50011, USA

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Abstract

The exact confidence interval for σ is hypersensitive to minor violations of the normality assumption and its performance does not improve with increasing sample size. An approximate confidence interval for σ is proposed and is shown to be nearly exact under normality with excellent small-sample properties under moderate nonnormality. The small-sample performance of the proposed interval may be further improved using prior kurtosis information. A sample size planning formula is given.

Keywords: Dispersion; Interval estimation; Kurtosis; Prior information; Sample size; Variability

1. Introduction

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample. If $Y_i \sim \mathrm{N}\left(\mu, \sigma^2\right)$ for all i, then an exact $100(1-\alpha)\%$ confidence interval for σ^2 is

$$(n-1)\hat{\sigma}^2/U < \sigma^2 < (n-1)\hat{\sigma}^2/L, \tag{1}$$

where $U = \chi^2_{\alpha/2;n-1}$, $L = \chi^2_{1-\alpha/2;n-1}$, $\hat{\sigma}^2 = \sum\left(Y_i - \hat{\mu}\right)^2/(n-1)$, $\hat{\mu} = \sum Y_i/n$, $\chi^2_{p,\mathrm{df}}$ is the point on a central chi-squares distribution with df degrees of freedom exceeded with

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* Tel.: +5152941975; fax: +5152944040.
E-mail address: dgbonett@iastate.edu (D.G. Bonett).

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probability p (Tate and Klett, 1959). Taking the square root of the endpoints of (1) gives a confidence interval for σ.

The exact confidence interval (1) is hypersensitive to minor violations of the normality assumption. The results of Scheffé (1959, p. 336) can be applied to show that (1) has an asymptotic coverage probability of about 0.76, 0.63, 0.60, and 0.51 for the Logistic, $t(7)$, Laplace, and $t(5)$ distributions, respectively. This result is disturbing because these symmetric distributions are not easily distinguished from a normal distribution unless the sample size is large. Miller (1986, p. 264) describes this situation as "catastrophic".

An alternative to the exact confidence interval is proposed here that: (1) is nearly exact under normality, (2) has coverage probability close to $1-\alpha$ under moderately nonnormality, (3) has coverage probability that approaches $1-\alpha$ as the sample size increases for nonnormal distributions with finite fourth moments, and (4) is not computationally intensive.

2. Proposed confidence interval

Instead of assuming $Y_i \sim \mathrm{N}\left(\mu, \sigma^2\right)$, let Y_i $(i = 1, 2, \ldots, n)$ be continuous, independent and identically distributed random variables with $0 < var\,(Y_i) = \sigma^2$, $E\,(Y_i) = \mu$ and finite fourth moment. The variance of $\hat{\sigma}^2$ may be expressed as $\sigma^4 \left\{\gamma_4 - (n-3)/(n-1)\right\}/n$, where $\gamma_4 = \mu^4/\sigma^4$ and μ^4 is the population fourth central moment (Mood et al., 1974, p. 229). A variance-stabilizing transformation for $\hat{\sigma}^2$ is $\ln\left(\hat{\sigma}^2\right)$ and application of the delta method gives $var\ \ln\left(\hat{\sigma}^2\right) \cong \left\{\gamma_4 - (n-3)/(n-1)\right\}/n$. Shoemaker, 2003 found that using $\left\{\gamma_4 - (n-3)/n\right\}/(n-1)$ improved the small-sample performance of his equal-variance test, and this small-sample adjustment will be used here. In practice, γ_4 is unknown and an estimate of $var\ \ln\left(\hat{\sigma}^2\right)$ will require an estimate of γ_4. Pearson's estimator $\hat{\gamma}_4 = n \sum \left(Y_i - \hat{\mu}\right)^4 / \left(\sum \left(Y_i - \hat{\mu}\right)^2\right)^2$ tends to have large negative bias in leptokurtic (heavy tailed) distributions unless the sample size is very large. The following estimator of γ_4, which is asymptotically equivalent to Pearson's estimator, is proposed

$$\bar{\gamma}_4 = n \sum (Y_i - m)^4 \Big/ \left(\sum \left(Y_i - \hat{\mu}\right)^2\right)^2, \tag{2}$$

where m is a trimmed mean with trim-proportion equal to $1/\left\{2(n-4)^{1/2}\right\}$ so that m converges to μ as n increases without bound. This estimator of kurtosis tends to have less negative bias and smaller coefficient of variability than Pearson's estimator in symmetric and skewed leptokurtic distributions.

In some applications a large-sample estimate of γ_4 from a previous study will be available. Let $\tilde{\gamma}_4$ denote a prior point estimate of γ_4 obtained from a sample of size n_0. The prior point estimate may be combined with (2) to give a pooled estimate of γ_4

$$\hat{\gamma}_4^* = \left(n_0\tilde{\gamma}_4 + n\bar{\gamma}_4\right)/(n_0 + n), \tag{3}$$

which obviously simplifies to (2) when prior information is unavailable.

A prior point estimate of γ_4 need not come from a single large sample but instead could be a pooled estimate from several small samples. When pooling kurtosis estimates from