Computational Statistics and Data Analysis 55 (2011) 1118–1136

Contents lists available at ScienceDirect

Computational Statistics and Data Analysis

journal homepage: www.elsevier.com/locate/csda

The beta Burr XII distribution with application to lifetime data

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ARTICLE INFO

Article history:
Received 30 October 2009
Received in revised form 4 August 2010
Accepted 9 September 2010
Available online 17 September 2010

Keywords:
Beta distribution
Burr XII distribution
Maximum likelihood
Observed information matrix
Order statistic
Weibull distribution

ABSTRACT

For the first time, a five-parameter distribution, the so-called beta Burr XII distribution, is defined and investigated. The new distribution contains as special sub-models some well-known distributions discussed in the literature, such as the logistic, Weibull and Burr XII distributions, among several others. We derive its moment generating function. We obtain, as a special case, the moment generating function of the Burr XII distribution, which seems to be a new result. Moments, mean deviations, Bonferroni and Lorenz curves and reliability are provided. We derive two representations for the moments of the order statistics. The method of maximum likelihood and a Bayesian analysis are proposed for estimating the model parameters. The observed information matrix is obtained. For different parameter settings and sample sizes, various simulation studies are performed and compared in order to study the performance of the new distribution. An application to real data demonstrates that the new distribution can provide a better fit than other classical models. We hope that this generalization may attract wider applications in reliability, biology and lifetime data analysis.

1. Introduction

The Burr system of distributions was constructed in 1942 by Irving W. Burr (see, Burr, 1942). Since the corresponding density functions have a wide variety of shapes, this system is useful for approximating histograms, particularly when a simple mathematical structure for the fitted cumulative distribution function (cdf) is required. Other applications include simulation, quantal response, approximation of distributions and development of non-normal control charts. A number of standard theoretical distributions are limiting forms of Burr distributions. Rodriguez (1977) showed that the Burr coverage area on a specific plane is occupied by various well-known and useful distributions, including the normal, log-normal, gamma, logistic and extreme-value type I distributions. The Burr XII (BXII) distribution, having the logistic and Weibull forms as special sub-models, is a very popular distribution for modeling lifetime data and for modeling phenomena with monotone failure rates. When modeling monotone hazard rates, the Weibull distribution may be an initial choice because of its negatively and positively skewed density shapes. However, it does not provide a reasonable parametric fit for modeling phenomena with non-monotone failure rates such as the bathtub-shaped and the unimodal failure rates that are common in reliability and biological studies. Such bathtub hazard curves have nearly flat middle portions and the corresponding densities have a positive anti-mode. Unimodal failure rates can be observed in the course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

The cdf and the reliability function of the BXII can be written in closed form; thus this simplifies the computation of the percentiles and the likelihood function for censored data. This distribution has algebraic tails which are effective for modeling failures that occur with lower frequency than with those models based on exponential tails. Hence, it represents a good

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doi:10.1016/j.csda.2010.09.009

model for modeling failure time data (Zimmer et al., 1998). Shao (2004) discussed the maximum likelihood estimation for the three-parameter BXII distribution. Shao et al. (2004) studied models for extremes using the extended three-parameter BXII distribution with application to flood frequency analysis. According to Soliman (2005), this distribution covers the curve shape characteristics for a large number of distributions. The versatility and flexibility of the BXII distribution makes it quite attractive as a tentative model for data whose underlying distribution is unknown. Wu et al. (2007) studied the estimation problems for this distribution on the basis of progressive type II censoring with random removals, where the number of units removed at each failure time has a discrete uniform distribution. Recently, Silva et al. (2008) proposed a location–scale regression model based on this distribution, referred to as the log-Burr XII regression model, for lifetime data analysis as a feasible alternative to the log-logistic regression model, and Silva et al. (2010b) proposed a residual for the log-Burr XII regression model whose empirical distribution is close to normality.

Many models have been introduced in the literature by extending the Weibull and exponential distributions. For example, the exponentiated Weibull (EW) (Mudholkar et al., 1995, 1996), the additive Weibull (Xie and Lai, 1995), the extended Weibull (Xie et al., 2002), the modified Weibull (MW) (Lai et al., 2003), the beta exponential (BE) (Nadarajah and Kotz, 2005), the beta Weibull (BW) (Lee et al., 2007), the extended flexible Weibull (Bebbington et al., 2007), the generalized modified Weibull (GMW) (Carrasco et al., 2008), the generalized inverse Weibull (Gusmão et al., 2009) and the beta modified Weibull (Silva et al., 2010a) distributions.

In this article, we propose a new distribution with five parameters, referred to as the beta Burr XII (BBXII) distribution, which contains as special sub-models the beta log-logistic (BLL), BW, BXII, Weibull and log-logistic (LL) distributions, among others. Our distribution due to its flexibility in accommodating different forms of the risk function is an important model that can be used in a variety of problems in modeling lifetime data. The BBXII distribution is not only convenient for modeling comfortable unimodal-shaped failure rates but it is also suitable for testing the goodness-of-fit of its special sub-models.

The rest of the paper is organized as follows. In Section 2, we define the BBXII distribution and present some special sub-models. In Section 3, we derive expansions for its probability density function (pdf) and cdf. We show that the density function of the BBXII distribution can be expressed as a mixture of BXII density functions. General expansions for the moments are also given in this section. In Section 4, we give an expansion for the moment generating function (mgf). In Section 5, we derive the mean deviations, the Bonferroni and Lorenz curves and the reliability. Section 6 is devoted to order statistics, their moments and *L*-moments (Hosking, 1986). In Section 7, we discuss maximum likelihood estimation and calculate the elements of the observed information matrix. In Section 8, we performed a simulation study for some parameter values. In Section 9, a Bayesian methodology was applied for the BBXII model. In Section 10, we provide an application of the new distribution to real data to show that it can yield a better fit than some other known models. Section 11 ends with some concluding remarks.

2. The model definition

With $G(x)$ denoting the baseline cdf, Eugene et al. (2002) defined a generalized class of distributions for $a > 0$ and $b > 0$ by

$$F(x) = I_{G(x)}(a, b) = \frac{1}{B(a, b)} \int_0^{G(x)} w^{a-1}(1-w)^{b-1}\mathrm{d}w, \tag{1}$$

where a and b are shape parameters. Here, $I_y(a, b) = B_y(a, b)/B(a, b)$ is the incomplete beta function ratio, $B_y(a, b) = \int_0^y w^{a-1}(1-w)^{b-1}\mathrm{d}w$ is the incomplete beta function and $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function, where $\Gamma(\cdot)$ is the gamma function. This class of generalized distributions has been receiving considerable attention over the last few years, in particular following the work of Jones (2004). In fact, on the basis of Eq. (1), Eugene et al. (2002), Nadarajah and Kotz (2004), Nadarajah and Gupta (2004) and Nadarajah and Kotz (2005) proposed the beta normal, beta Gumbel, beta Fréchet and beta exponential distributions, respectively. More recently, Barreto-Souza et al. (2010) and Pescim et al. (2010) introduced the beta generalized exponential and beta generalized half-normal distributions, respectively. Another distribution that happens to belong to Eq. (1) is the beta logistic distribution, which has been around for over 20 years (Brown et al., 2002), even if it did not originate directly from this equation.

The density function corresponding to (1) is given by

$$f(x) = \frac{g(x)}{B(a, b)} G(x)^{a-1}\{1 - G(x)\}^{b-1}, \tag{2}$$

where $g(x) = \mathrm{d}G(x)/\mathrm{d}x$ is the density function of the baseline distribution. The density $f(x)$ will be most tractable when the functions $G(x)$ and $g(x)$ have simple analytic expressions as in the cases of the BXII and Weibull distributions. Except for some special choices for $G(x)$, Eq. (2) will be difficult to deal with in generality.

We are motivated to introduce the BBXII distribution because of the generalizations discussed in Section 1, the wide usage of the BXII distribution and the fact that the current generalization provides the means for its continuous extension to still more complex situations. Zimmer et al. (1998) introduced the three-parameter BXII distribution with the cdf and pdf (for $x > 0$) given by

$$G(x; s, k, c) = 1 - \left[1 + \left(\frac{x}{s}\right)^c\right]^{-k} \tag{3}$$