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Testing on the common mean of several normal distributions

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ABSTRACT

Point estimation of the common mean of several normal distributions with unknown and possibly unequal variances has attracted the attention of many researchers over the last five decades. Relatively less attention has been paid to the hypothesis testing problem, presumably due to the complicated sampling distribution(s) of the test statistics(s) involved. Taking advantage of the computational resources available nowadays there has been a renewed interest in this problem, and a few test procedures have been proposed lately including those based on the generalized *p*-value approach. In this paper we propose three new tests based on the famous Graybill–Deal estimator (GDE) as well as the maximum likelihood estimator (MLE) of the common mean, and these test procedures appear to work as good as (if not better than) the existing test methods. The two tests based on the GDE use respectively a first order unbiased variance estimate proposed by Sinha [Sinha, B.K., 1985. Unbiased estimation of the variance of the Graybill–Deal estimator of the common mean of several normal populations. The Canadian Journal of Statistics 13 (3), 243–247], as well as the little known exact unbiased variance estimator proposed by Nikulin and Voinov [Nikulin, M.S., Voinov, V.G., 1995. On the problem of the means of weighted normal populations. Qüestiió (Quaderns d'Estadistica, Sistemes, Informatica i Investigació Operativa) 19 (1–3), 93–106] (after we've fixed a small mistake in the final expression). On the other hand, the MLE, which doesn't have a closed expression, uses a parametric bootstrap method proposed by Pal, Lim and Ling [Pal, N., Lim, W.K., Ling, C.H., 2007b. A computational approach to statistical inferences. Journal of Applied Probability & Statistics 2 (1), 13–35]. The extensive simulation results presented in this paper complement the recent studies undertaken by Krishnamoorthy and Lu [Krishnamoorthy, K., Lu, Y., 2003. Inferences on the common mean of several normal populations based on the generalized variable method. Biometrics 59, 237–247], and Lin and Lee [Lin, S.H., Lee, J.C., 2005. Generalized inferences on the common mean of several normal populations. Journal of Statistical Planning and Inference 134, 568–582].

1. Introduction

One of the oldest and most interesting problems in statistical sciences is the inference on a common mean of several normal distributions with unknown and possibly unequal variances. In literature, this is also known as Meta-Analysis where data from multiple sources are combined or integrated with a common objective. This problem arises in situations where different instruments or methods are used repeatedly to measure, say, blood alcohol level, or lead content in gasoline. One wishes to use multiple datasets, assuming normality, for an improved inference of the common mean, rather than relying on individual samples. As an application of the Meta-Analysis in clinical trials see Vazquez et al. (2007).

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To formulate the present problem, assume that we have *iid* observations $X_{i1}, \ldots, X_{in_i}$ from $N(\mu, \sigma_i^2)$, $1 \le i \le k$, where all parameters are assumed to be unknown. Define $\bar{X}_i$ and S_i as

$$\bar{X}_i = \sum_{j=1}^{n_i} X_{ij}/n_i, \qquad S_i = \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2; \tag{1.1}$$

where $\bar{X}_i \sim N(\mu, \sigma_i^2/n_i)$, $S_i \sim \sigma_i^2 \chi^2_{(n_i-1)}$ $(1 \le i \le k)$, and these statistics are all mutually independent. Throughout this paper it is assumed that $n_i \ge 2$ $(1 \le i \le k)$ unless mentioned otherwise.

Inference on the common mean μ has its genesis in a balanced incomplete block design (BIBD) with uncorrelated random block effects. For the lth treatment effect (say, τ_ℓ) one has two point estimates — namely, the intra-block estimate and the interblock estimate (say, $\hat{\tau}_\ell$ and $\hat{\tau}_\ell^*$ respectively). Under the usual design assumptions, $\hat{\tau}_\ell$ and $\hat{\tau}_\ell^*$ are independent, have normal distributions with common mean τ_ℓ, but with unknown and possibly unequal variances. Hence we have a common mean problem with $k = 2$ (see also Montgomery (1997), page-216–218). Apart from estimating the lth treatment effect, one may wish to verify whether it is 0 or not, and thereby getting into a hypothesis testing problem. Zacks (1966, 1970), who contributed significantly to the theoretical foundation of the above mentioned common mean problem, described how he got interested in this problem by an application in soil science (see Kempthorne et al. (1991)).

Point estimation of μ has drawn the attention of many researchers over the past five decades. When σ_i's are all known, the optimal estimator (MLE, BLUE as well as UMVUE) is

$$\hat{\mu} = \sum_{i=1}^{k} (n_i/\sigma_i^2)\bar{X}_i \Big/ \sum_{i=1}^{k} (n_i/\sigma_i^2). \tag{1.2}$$

But when σ_i's are all unknown, one encounters the real challenge to propose an efficient estimator of μ combining the individual sample means $\bar{X}_i$'s.

Note that the minimal sufficient statistic $(\bar{X}_i, S_i, i = 1, 2, \ldots, k)$ is not complete. As a result, one can not get the UMVUE (if it exists) using the standard Rao–Blackwell theorem on an unbiased estimator for estimating μ.

In our present problem, where σ_i's are all unknown, the most appealing unbiased estimator of μ has been the Graybill–Deal estimator (GDE) given as

$$\hat{\mu}_{\mathrm{GDE}} = \sum_{i=1}^{k} (n_i/s_i^2)\bar{X}_i \Big/ \sum_{i=1}^{k} (n_i/s_i^2), \tag{1.3}$$

where $s_i^2 = S_i/(n_i - 1)$, $1 \le i \le k$. Graybill and Deal (1959) obtained conditions on the n_i's for $k = 2$ only for which $\hat{\mu}_{\mathrm{GDE}}$ has a smaller variance than each $\bar{X}_i$ $(i = 1, 2)$.

It is important to remember that for estimating μ the above $\hat{\mu}_{\mathrm{GDE}}$ is not the MLE. The MLE, which does not have any closed expression, is

$$\hat{\mu}_{\mathrm{MLE}} = \sum_{i=1}^{k} (n_i/\hat{\sigma}^2_{i(\mathrm{MLE})})\bar{X}_i \Big/ \sum_{i=1}^{k} (n_i/\hat{\sigma}^2_{i(\mathrm{MLE})}), \tag{1.4}$$

where $\hat{\sigma}^2_{i(\mathrm{MLE})}$ is the MLE of σ_i^2 $(1 \le i \le k)$ obtained by solving the following system of equations:

$$\hat{\sigma}^2_{i(\mathrm{MLE})} = (S_i/n_i) + \left[\left\{ \sum_{j=1}^{k} n_j(\bar{X}_i - \bar{X}_j)/\hat{\sigma}^2_{j(MLE)} \right\} \Big/ \left\{ \sum_{j=1}^{k} n_j/\hat{\sigma}^2_{j(MLE)} \right\} \right]^2, \quad i = 1, 2, \ldots, k. \tag{1.5}$$

It is because of this complicated expression that the MLE has dampened the interest of many researchers. In a recent paper Pal et al. (2007a) compared the MLE with the GDE for $k = 2$, and observed that the MLE can be advantageous in a heavily unbalanced case. For a comprehensive review on the point estimation of μ till 1990 one can refer to Kubokawa (1991).

The main objective of this paper is to consider testing

$$H_0 : \mu = \mu_0 \text{ vs. } H_a : \mu \ne \mu_0 \tag{1.6}$$

based on the statistics in (1.1). For a better understanding, we may occasionally focus on the special case of $k = 2$. This special case is easy to understand in the context of BIBD mentioned earlier, but can be extended suitably for general k. Cohen and Sackrowitz (1977) showed that each individual t-test based only on one sample is admissible (which has the highest power at some boundary region of the parameter space). This is somewhat surprising in the light of inadmissibility results obtained in point estimation where, as expected, the combined data is supposed to help in inferences pertaining to the common mean. For the special case $k = 2$ and $n_1 = n_2 = n$, Cohen and Sackrowitz (1984) considered several