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Applications of the characteristic function-based continuum GMM in finance

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ABSTRACT

A review of the theoretical properties of the GMM with a continuum of moment conditions is presented. Numerical methods for its implementation are discussed. A simulation study based on the stable distribution and an empirical application based on the autoregressive variance Gamma model are performed. Using the Alcoa price data, the findings suggest that investors require a positive premium for bearing the expected risk while a negative penalty is attached to unexpected risk.

1. Introduction

For many interesting financial econometric models, the characteristic function (CF) is available in closed form while the likelihood function is not, for example stable distributions and discretely sampled continuous time processes. Exceptionally, a discrete sample from a square root diffusion model admits a closed form conditional likelihood expression. Unfortunately, this expression takes the form of an infinite sum that must be truncated in practice. Certain discrete time models (e.g. the variance gamma model) also have known closed form likelihood functions that are not convenient for numerical optimization. In these situations, the use of the CF for inference is an attractive alternative. In fact, two random variables have the same distribution if and only if their CF coincides on the whole real line. This suggests that an inference method that adequately exploits the information content of the CF has the potential to achieve the same level of efficiency as a likelihood-based approach. One such inference method proposed by Carrasco and Florens (2000) for IID models exploits the whole continuum of moment conditions based on the difference between the empirical and theoretical characteristic functions. Carrasco et al. (2007a) extend the method to deal with Markov and dependent models. Other leading works in this area include Singleton (2001), Knight and Yu (2002), Knight et al. (2002) and Chacko and Viceira (2003). A good review of this literature is provided by Yu (2004).

The goal of this paper is to make the Generalized Method of Moments with a continuum of moments conditions (CGMM) based on the CF accessible to applied researchers. Our focus will be on the approach proposed by Carrasco and Florens (2000) and its extension by Carrasco et al. (2007a). First of all, we review the theory underlying the CGMM. We recall the main assumptions that are useful for the consistency and asymptotic normality of the CGMM estimator. Next, we discuss in detail the important steps of the implementation of the CGMM in practice. Finally, we provide a simulation study with the stable distribution and an empirical application with the autoregressive variance gamma model.

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The stable distribution has been introduced in finance to fit the asymmetry and fat tails observed empirically in the distributions of assets returns (Mandelbrot, 1963, or McCulloch, 1986). In its common parameterization, it has a stability parameter $\alpha \in (0, 2]$, a skewness parameter $\beta \in [-1, 1]$, a scale parameter $\sigma > 0$ and a location parameter $\mu \in \mathbb{R}$. The moments of order larger than α do not exist for the stable distribution when $\alpha < 2$. When $\alpha = 2$, all the moments exist but the asymmetry parameter β is no longer identifiable. Closed form expressions for stable densities are available only in a few cases. For example, the case $\alpha = 2$ reduces to a normal distribution $N(\alpha, 2\sigma_0^2)$. When $\alpha = 1$ and $\beta_0 = 0$, we obtain the Cauchy distribution whereas the case $\alpha = 1/2$ and $\beta_0 = 1$ results in the so-called Levy distribution. An identity established by Zolotarev (1986) and commented in Weron (1996) allows one to deduce the density of the case $\alpha = 1/2$ and $\beta_0 = -1$ from the previous one. But as pointed out by Nolan (in preparation), the knowledge of the likelihood function at isolated values of the parameter space is not helpful when one is trying to fit the model to real data. This difficulty has often led researchers to rely on numerical approximations of the likelihood of the stable distribution. For example, McCulloch (1998) discusses an approximate maximum likelihood procedure for symmetric stable distributions while Nolan (1997) proposes alternative numerical procedures for $\alpha > 0.1$. Mittnik et al. (1999) and Paolella (2007) propose Fast Fourier Transform algorithms to approximate the likelihood function. An alternative quantile based approach is also discussed in McCulloch (1986).

In the current paper, a CGMM procedure that can be used without imposing any restriction on the parameter space is presented. Monte Carlo simulations show that the CGMM outperforms the standard GMM that uses a finite number of moment conditions based on the CF. However, the variance of the estimators cannot be computed analytically when the vector of parameters is close to the non-identification region (that is, when α is close to 2). One then has to rely on Monte Carlo simulations to build confidence intervals. As the primary goal of the current paper is to illustrate the implementation of the CGMM, we leave the simulation comparison of the latter with the likelihood-based approaches for future investigations.

The fact that the asymmetry and fat-tailedness of the stable distribution vanish when its variance exists ($\alpha = 2$) limits its use for the purpose of modeling assets returns. A simple way to circumvent this limitation consists in modeling the variance of the returns as a Gamma variable. This yields the variance gamma models. The symmetric variance gamma model has been proposed by Madan and Senata (1990). Madan et al. (1998) extend the basic model to include asymmetry. These two models unfortunately assume that the variance is IID. Here we relax this assumption by assuming that the variance follows the autoregressive gamma process studied in Gourieroux and Jasiak (2005). The resulting model for assets returns is termed the *"autoregressive variance gamma model"*. We propose an estimation strategy in two steps. In the first step, we fit the autoregressive gamma model to a consistent estimator of the daily integrated variance used as a proxy for the true daily variance. Next, we estimate a relationship between returns and volatility that allows us to disentangle the risk premium from the leverage effect. An empirical application with the Alcoa stock listed in the Dow Jones Industrials shows that investors require a positive premium for bearing expected risk while a negative premium is attached to unexpected risk.

The rest of the paper is organized as follows. The next section reviews the main theoretical results on the CGMM. In Section 3, we discuss the numerical aspects of its implementation. In Section 4, we present a simulation study of the performance of the CGMM to estimate the stable distribution. In Section 5, we present and estimate the autoregressive variance gamma model both with simulated and real data. Section 6 concludes. Some technical derivations are left in the Appendix.

2. The CGMM: A brief theoretical review

In this section, we present the theoretical framework underlying the CGMM estimation. The first subsection reviews the IID framework while the second subsection deals with the dependent case. In the third subsection, we discuss the assumptions needed for the CGMM estimator to have good asymptotic properties.

2.1. *The CGMM in the* IID *case*

Let $(x_1, \ldots, x_T)$ be an IID sample of an m-dimensional vector process whose CF is given by $E^{\theta_0}(e^{i\tau' x_t}) = \varphi(\tau, \theta_0)$, where θ_0 is a finite dimensional parameter that fully characterizes the distribution of $\{x_t\}$ and $\tau \in \mathbb{R}^m$ is the Fourier transformation variable. By definition of $\varphi(\tau, \theta_0)$, the following set of moment functions can be considered for the purpose of estimating the parameter θ_0:

$$h_t(\tau, \theta_0) = e^{i\tau' x_t} - \varphi(\tau, \theta_0), \quad \text{for all } \tau \in \mathbb{R}^m. \tag{1}$$

Note that these moment functions are indexed by $\tau \in \mathbb{R}^m$, and hence we have a continuum of moment conditions. Since the CF contains the same information as the likelihood function, an efficient use of the whole continuum of moment conditions permits us to achieve the maximum likelihood efficiency (see Carrasco and Florens, 2000).

As in Feuerverger and McDunnough (1981b), Singleton (2001) or Chacko and Viceira (2003), one may choose to estimate θ_0 using GMM based on a discrete subset of the continuum (1). More precisely, let $\{h_t(\tau_k, \theta_0)\}_{k=1}^{q}$ be a discrete subset of q moments conditions drawn from (1), and define the vector $g_t(\theta_0)$ by:

$$g_t(\theta_0) = (\text{Re}\, h_t(\tau_1, \theta_0), \ldots, \text{Re}\, h_t(\tau_q, \theta_0), \text{Im}\, h_t(\tau_1, \theta_0), \ldots, \text{Im}\, h_t(\tau_q, \theta_0))'.$$

The standard GMM estimator of θ_0 is computed as:

$$\widehat{\theta}_{\text{GMM}} = \arg\min_{\theta} \widehat{g}(\theta_0)' \widehat{S}^{-1} \widehat{g}(\theta_0),$$