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Recursive computation of piecewise constant volatilities[☆]

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ABSTRACT

Returns of risky assets are often modelled as the product of a volatility function and standard Gaussian white noise. Long range data cannot be adequately approximated by simple parametric models. The choice is between retaining simple models and segmenting the data, or to use a non-parametric approach. There is not always a clear dividing line between the two approaches. In particular, modelling the volatility as a piecewise constant function can be interpreted either as segmentation based on the simple model of constant volatility, or as an approximation to the observed volatility by a simple function. A precise concept of local approximation is introduced and it is shown that the sparsity problem of minimizing the number of intervals of constancy under constraints can be solved using dynamic programming. The method is applied to the daily returns of the German DAX index. In a short simulation study it is shown that the method can accurately estimate the number of breaks for simulated data without prior knowledge of this number.

1. The problem

Let $R(t)$ be the return of some risky asset in period t. For stocks with end of period price P_t, $R(t) = \ln(P_t/P_{t-1})$. In empirical finance, $R(t)$ is often decomposed as

$$R(t) = \sigma(t) \cdot Z(t), \quad t = 1, \ldots, n, \tag{1}$$

where Z is standard Gaussian white noise. The method can be adapted to other distributional assumptions such as in Curto et al. (2009). This will be briefly discussed in Section 6. The major problem is how best to model $\sigma(t)$. In the enormous ARCH-class of models, for example, $\sigma(t)$ depends on past values of the $R(t)^2$ while unconditional volatilities remain constant over time. On the other hand, there seems to emerge a consensus in empirical finance, to be explored in the present paper, that unconditional volatilities do vary and are best modelled, in the absence of external information such as in Wilfling (2003), as piecewise constant functions of time (Mercurio and Spokoiny, 2004; Granger and Stărică, 2005).

This can of course be done in various ways. The approach taken below is a non-parametric one in the line of Davies and Kovac (2001), Vassiliou and Demetriou (2005), Davies (2005, 2006) and Davies et al. (2009). The goal is to give a sparse piecewise constant approximation to the volatility. It is not assumed that the underlying volatility is piecewise constant with the goal of identifying the breaks although the method will identify breaks in the volatility if they are sufficiently pronounced. If the volatility is continuously increasing over a period of time and cannot be well approximated by a constant

[☆] The algorithms suggested here were programmed in R and are available from the authors upon request.

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volatility the piecewise constant approximation will have breaks which do not correspond to (the non-existent) breaks in the underlying volatility. Not all breaks in the piecewise constant approximation will therefore necessarily correspond to breaks in the underlying volatility although some will. The mathematics of a piecewise constant approximation to a continuous function and the associated theory of non-parametric regression are given in Boysen et al. (2009).

The method exploits the fact that, under the model (1)

$$\sum_{t\in I} \frac{R(t)^2}{\sigma(t)^2} \sim \chi^2_{|I|} \tag{2}$$

for any nonempty interval $I \subset \{1, \ldots, n\}$ where $|I|$ denotes the number of elements of I. This implies that, for all $\alpha \in (0, 1)$, there exists $\alpha_n \in (0, 1)$ such that

$$P\left(\chi^2_{|I|,\frac{1-\alpha_n}{2}} \le \sum_{t\in I} \frac{R(t)^2}{\sigma(t)^2} \le \chi^2_{|I|,\frac{1+\alpha_n}{2}},\ \forall I \subset \{1,\ldots,n\}\right) = \alpha. \tag{3}$$

Let $\mathcal{A}_n(\alpha_n)$ denote the set of all functions $\sigma : \{1, \ldots, n\} \to \mathbb{R}^+$ such that the inequalities within the brackets of (3) hold. It may be checked that for $\alpha_n > 0.5$ (which will always be the case)

$$\chi^2_{|I|,\frac{1-\alpha_n}{2}} < |I| < \chi^2_{|I|,\frac{1+\alpha_n}{2}} \tag{4}$$

for all non-empty intervals $I \subset \{1, \ldots, n\}$. This implies that $\tilde{\sigma}(t) := |R(t)|$ lies in $\mathcal{A}_n(\alpha_n)$, which is consequently a nonempty set (ignoring the possibility that $|R(t)| = 0$). The problem then becomes one of sensibly choosing amongst the many possibilities.

In line with Davies (2005, 2006) a sparsity approach is taken. The problem is to determine those functions $\sigma(t)$ in $\mathcal{A}_n(\alpha_n)$ which are piecewise constant on intervals and minimize the number of intervals of constancy. This is a computational problem which is essentially unsolvable as it stands. A modified version based on local adequacy can be solved using dynamic programming. It is considered in detail below.

2. Minimizing the number of intervals

To define the modified problem let $I_1, \ldots, I_k \subset \{1, \ldots, n\}$, $I_\nu \cap I_\mu = \emptyset$ and $I_1 \cup \cdots \cup I_k = \{1, \ldots, n\}$, be the intervals where $\sigma(t)$ is constant, with value σ_{I_ν}, ($\sigma_{I_\nu} > 0$). The inequalities in Eq. (3) imply

$$\frac{\sum_{t\in I} R(t)^2}{\chi^2_{|I|,\frac{1+\alpha_n}{2}}} \le \sigma^2_{I_\nu} \le \frac{\sum_{t\in I} R(t)^2}{\chi^2_{|I|,\frac{1-\alpha_n}{2}}},\quad \forall I \subset I_\nu,\ \nu = 1,\ldots,k. \tag{5}$$

A volatility function which satisfies these constraints is called locally adequate. Local adequacy is a weaker condition than (3) and it turns out that the sparsity problem can be solved for piecewise constant locally adequate volatility functions. It follows from (5) that

$$\sigma_l^2(I_\nu) := \max_{J\subset I_\nu} \frac{\sum_{t\in J} R(t)^2}{\chi^2_{|J|,\frac{1+\alpha_n}{2}}} \le \sigma^2_{I_\nu} \le \min_{J\subset I_\nu} \frac{\sum_{t\in J} R(t)^2}{\chi^2_{|J|,\frac{1-\alpha_n}{2}}} =: \sigma_u^2(I_\nu). \tag{6}$$

Given the left endpoint s_ν of I_ν, the lower bound $\sigma_l^2(I_\nu)$ is an increasing function of the right endpoint t_ν, and the upper bound $\sigma_u^2(I_\nu)$ is a decreasing function of the right endpoint t_ν of I_ν. This suggests the following algorithm to obtain a locally adequate volatility function: start with $s_1 = 1$, $t_1 = 1$ and let t_1 increase until the upper bound becomes smaller than the lower bound at $t_1 + 1$. Setting $s_2 = t_1 + 1$ the process is repeated until the end of the sample is reached.

Theorem 1. *The volatility function σ constructed above is locally adequate and has the minimum number of intervals of constancy.*

Proof. Assume that there exists another locally adequate volatility function $\tilde{\sigma}$ with corresponding partition $\tilde{I}_1, \ldots, \tilde{I}_{\tilde{k}}$ with $\tilde{k} < k$. It is clear that $\tilde{I}_1 \subset I_1$ and by induction it follows that $\cup_{j=1}^{i} \tilde{I}_j \subset \cup_{j=1}^{i} I_j$. Consequently

$$\{1,\ldots,n\} = \bigcup_{j=1}^{\tilde{k}} \tilde{I}_j \subset \bigcup_{j=1}^{\tilde{k}} I_j \overset{\neq}{\subset} \{1,\ldots,n\} \tag{7}$$

which is a contradiction. □