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Exact optimal inference in regression models under heteroskedasticity and non-normality of unknown form[☆]

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ABSTRACT

Simple point-optimal sign-based tests are developed for inference on linear and nonlinear regression models with non-Gaussian heteroskedastic errors. The tests are exact, distribution-free, robust to heteroskedasticity of unknown form, and may be inverted to build confidence regions for the parameters of the regression function. Since point-optimal sign tests depend on the alternative hypothesis considered, an adaptive approach based on a split-sample technique is proposed in order to choose an alternative that brings power close to the power envelope. The performance of the proposed *quasi-point-optimal* sign tests with respect to size and power is assessed in a Monte Carlo study. The power of quasi-point-optimal sign tests is typically close to the power envelope, when approximately 10% of the sample is used to estimate the alternative and the remaining sample to compute the test statistic. Further, the proposed procedures perform much better than common least-squares-based tests which are supposed to be robust against heteroskedasticity.

1. Introduction

Regression errors in economic data frequently exhibit non-normal distributions and heteroskedasticity. In the presence of several types of heteroskedasticity, usual "robust" tests – such as tests based on White (1980)-type variance corrections – remain plagued by poor size control and/or low power. This is the case, in particular, when there is a break in the disturbance variance or with a GARCH structure with one or several outliers. Further, the available *exact parametric* tests typically assume Gaussian disturbances. The latter assumption is often unrealistic and, in the presence of heavy tails and asymmetric distributions, the associated tests may easily not perform well in terms of size control or power. Furthermore, statistical procedures for inference on parameters of *nonlinear* models are typically based on asymptotic approximations, which may easily not be reliable in finite samples (see Dufour, 2003).

The present paper proposes simple point-optimal sign-based tests in linear and nonlinear regression models, which are valid under non-normality and heteroskedasticity of unknown form, provided the errors have median zero conditional on

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the explanatory variables. The proposed tests are exact, distribution-free, robust against heteroskedasticity of unknown form, and may be inverted to build confidence regions for the vector of unknown parameters. The setup and the type of procedures we consider are motivated in at least two ways.

First, it is well known that hypotheses on means (or moments) are not testable in nonparametric setups even under the apparently restrictive assumption that the observations are independent and identically distributed (*i.i.d.*): if a test has level α for testing the null hypothesis that the mean of *i.i.d.* observations has a given value, then its power cannot be larger than the level α under any alternative of the mean; see Bahadur and Savage (1956). Similar results hold for coefficients of regression models; see Dufour et al. (2008). In other words, moments are not empirically meaningful in many common nonparametric models. This provides a strong reason for focusing on quantile parameters (such as medians) in nonparametric models – instead of moments – because quantiles are not affected by such problems of non-testability.

Second, in the presence of general heteroskedasticity, Lehmann and Stein (1949) and Pratt and Gibbons (1981) show that sign methods are the only possible way of producing valid inference in finite samples; see also Dufour and Hallin (1991) and Dufour (2003). If a test has level α for testing the null hypothesis that the observations are independent each one with a distribution symmetric about zero, then its level must be equal to α conditional on the absolute values of the observations: in other words, it must be a *sign test*. For a more detailed discussion of statistical inference impossibilities in nonparametric models, see Dufour (2003) and Dufour et al. (2008).

A number of sign-based test procedures have been developed in the literature. In the presence of only one explanatory variable, Campbell and Dufour (1991, 1995, 1997) propose nonparametric analogues of the *t-test*, based on sign and signed rank statistics, which are applicable when regressors involve feedback of the type considered by Mankiw and Shapiro (1986). These tests are exact even when the disturbances are asymmetric, non-normal, and heteroskedastic. Boldin et al. (1997) propose locally optimal sign-based inference and estimation for linear models. Coudin and Dufour (2009) extend the work by Boldin et al. (1997) to account for serial dependence and discrete distributions. Wright (2000) proposes variance-ratio tests based on signs and ranks to test the null hypothesis that the series of interest is a martingale difference sequence. For other sign-based test procedures, the reader can consult Capanu et al. (2006) and Gerard and Schucany (2007) among others.

The present paper focuses on the optimality of sign tests and derives point-optimal tests based on sign statistics. Point-optimal tests are useful in a number of ways and they are particularly attractive when testing an economic theory against another one. An important feature of these tests comes from the fact that they trace out the *power envelope*, i.e. the maximum achievable power for a given testing problem. The power envelope provides an obvious benchmark against which test procedures can be evaluated. An early review and discussion of point-optimal tests is available in King (1987–88). More recently, this technique has been exploited in several papers in order to improve power. Dufour and King (1991) use point-optimal tests to do inference on the autocorrelation coefficient of a linear regression model with first-order autoregressive normal disturbances. Elliott et al. (1996) derive the asymptotic power envelope for point-optimal tests of a unit root in the autoregressive representation of a Gaussian time series under various trend specifications. Jansson (2005) derives an asymptotic Gaussian power envelope for tests of cointegration and proposes a feasible point-optimal cointegration test whose local asymptotic power function is found to be close to the asymptotic Gaussian power envelope. Begum and King (2005) propose a new approach for testing a composite null against a composite alternative hypothesis based on the generalized Neyman–Pearson lemma and maximize average power subject to controlling average size over different subsets of the null hypothesis parameter space. Liang et al. (2008) suggest locally optimal tests for exponential distributions with type-I censoring.

Since point-optimal sign (hereafter POS) tests depend on the alternative hypothesis, we propose an adaptive approach based on a split-sample technique (Dufour and Torrès, 1998; Dufour and Jasiak, 2001) to choose an alternative that makes the power curve of the POS test close to the power envelope. The idea consists in dividing the sample into two independent parts and using the first one to estimate the value of the alternative hypothesis and the second to compute the POS test statistic (Dufour and Taamouti, 2003; Dufour and Iglesias, 2008). The simulation results show that using approximately 10% of the sample to estimate the alternative yields a power function which is typically very close to the power envelope. We present a Monte Carlo study assessing the performance of the proposed "quasi-POS" test by comparing its size and power to those of some common tests which are supposed to be robust against heteroskedasticity. The results show that our procedures work quite well.

The plan of the paper is as follows. In Section 2, we present a general framework for deriving POS tests. In Section 3, we propose POS tests in the context of linear and nonlinear regression models. In Section 4, we study the power properties of the POS tests and propose an adaptive approach to choose an optimal alternative. In Section 5, we discuss the construction of the POS confidence regions using projection techniques. In Section 6, we present a Monte Carlo study assessing the performance of POS tests by comparing their size and power to those of some popular tests. We conclude in Section 7. Proofs are presented in Appendix A.

2. General framework

In this section, we describe a framework for deriving POS tests in the context of a general hypothesis testing problem. Point-optimal tests are useful in a number of ways and they are most attractive for problems in which the parameter space