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Inference for a simple step-stress model with competing risks for failure from the exponential distribution under time constraint

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ABSTRACT

In reliability analysis, accelerated life-testing allows for gradual increment of stress levels on test units during an experiment. In a special class of accelerated life tests known as step-stress tests, the stress levels increase discretely at pre-fixed time points, and this allows the experimenter to obtain information on the parameters of the lifetime distributions more quickly than under normal operating conditions. Moreover, when a test unit fails, there are often more than one fatal cause for the failure, such as mechanical or electrical. In this article, we consider the simple step-stress model under time constraint when the lifetime distributions of the different risk factors are independently exponentially distributed. Under this setup, we derive the maximum likelihood estimators (MLEs) of the unknown mean parameters of the different causes under the assumption of a cumulative exposure model. Since it is found that the MLEs do not exist when there is no failure by any particular risk factor within the specified time frame, the exact sampling distributions of the MLEs are derived through the use of conditional moment generating functions. Using these exact distributions as well as the asymptotic distributions, the parametric bootstrap method, and the Bayesian posterior distribution, we discuss the construction of confidence intervals and credible intervals for the parameters. Their performance is assessed through Monte Carlo simulations and finally, we illustrate the methods of inference discussed here with an example.

1. Introduction

In order to guarantee the service life and performance of a product, or even to compare alternative manufacturing designs, life-testing under normal operating conditions is obviously most desirable. However, due to the continual improvement in manufacturing design and technology, one often experiences difficulty in obtaining sufficient information about the failure time distribution of the products. As the products become highly reliable with substantially long life-spans, time-consuming and expensive tests are often required to collect enough failure data, which are necessary to draw inference about the relationship of lifetime with the external stress variables. In such situations, the standard life-testing methods are not appropriate, especially when developing prototypes of new products. This difficulty is overcome by accelerated life tests wherein the units are subjected to higher stress levels in order to cause rapid failures. Accelerated life tests allow the experimenter to apply more severe stresses to obtain information on the parameters of the lifetime distributions more quickly than would be possible under normal operating conditions. Some key references in the field of the accelerated life-testing include Nelson and Meeker (1978), Nelson (1990), Meeker and Escobar (1998) and Bagdonavicius and Nikulin (2002).

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A special class of the accelerated life-testing, known as step-stress testing, allows the experimenter to gradually increase the stress levels at some pre-fixed time points during the experiment for maximal flexibility and adjustability. This model has attracted great attention in the reliability literature. Sedyakin (1966) proposed one of the fundamental models in this area, known as the *cumulative damage* or *cumulative exposure* model. This model has been further discussed and generalized by Bagdonavicius (1978) and Nelson (1980). Miller and Nelson (1983), Gouno et al. (2004), Han et al. (2006), Balakrishnan et al. (2007), Balakrishnan and Xie (2007a,b) Balakrishnan et al. (2009) and Balakrishnan and Han (2009) have all discussed different inferential issues regarding the accelerated life-testing under the assumption of the cumulative exposure model. For a concise review of step-stress models, readers are referred to Gouno and Balakrishnan (2001) and Balakrishnan (2009).

Furthermore, in reliability analysis, it is common that a failure is associated with one of several fatal risk factors the test unit is exposed to. Since it is not usually possible to study the test units with an isolated risk factor, it becomes necessary to assess each risk factor in the presence of other risk factors. In order to analyze such a competing risks model, each failure observation must come in a bivariate form composed of a failure time and the cause of failure. Cox (1959), David and Moeschberger (1978), Klein and Basu (1981, 1982), and Crowder (2001) have all investigated the competing risks models and considered some specific parametric lifetime distributions for each risk factor. In addition to multiple causes of failure, censoring is also common in reliability experiments for many reasons such as time constraint and cost reduction. Among different censoring schemes, the conventional Type-I right censoring corresponds to the situation when the experiment gets terminated at a pre-fixed time point.

In this paper, we consider the simple step-stress model (*i.e.*, two stress levels) under time constraint (*i.e.*, Type-I censoring) when the lifetime distributions of the different risk factors are independently exponentially distributed. In Section 2, we present the MLEs of the mean parameters of the different risk factors and show that these MLEs do not always exist. The conditional MLEs are therefore proposed and the exact conditional distributions of these MLEs are derived in Section 3. Based on the exact distributions of the MLEs, we propose exact confidence intervals for the unknown mean parameters in Section 4. We also present the asymptotic distributions of the MLEs and the corresponding asymptotic confidence intervals as well as the confidence intervals from a parametric bootstrap method and the credible intervals from a Bayesian viewpoint. In Section 5, the performance of these confidence/credible intervals is evaluated in terms of probability coverages via Monte Carlo simulations. In Section 6, we present a numerical example to illustrate all the methods of inference developed in this article, and some concluding remarks are finally made in Section 7.

2. Model description and MLEs

A random sample of n identical units is placed on a life test under the initial stress level s_1. The successive failure times are then recorded along with the information about which risk factor caused each failure. At a pre-fixed time τ, the stress level is increased to s_2 and the life test continues until a pre-specified censoring time τ_c $(>\tau)$. When all n units fail before τ_c, then a complete set of failure observations would result for this simple step-stress test (*i.e.*, no censoring). Suppose each unit fails by one of two fatal risk factors and the time-to-failure by each competing risk has an independent exponential distribution which obeys the cumulative exposure model. Let θ_{ij} be the mean time-to-failure of a test unit at the stress level s_i by the risk factor j for $i, j = 1, 2$. Then, the cumulative distribution function (CDF) of the lifetime T_j due to the risk factor j is given by

$$G_j(t) = G_j(t; \theta_{1j}, \theta_{2j}) = \begin{cases} 1 - \exp\left\{-\dfrac{1}{\theta_{1j}} t\right\} & \text{if } 0 < t < \tau \\ 1 - \exp\left\{-\dfrac{1}{\theta_{1j}}\tau - \dfrac{1}{\theta_{2j}}(t-\tau)\right\} & \text{if } \tau \le t < \infty \end{cases} \tag{2.1}$$

for $j = 1, 2$, and the corresponding probability density function (PDF) of T_j is given by

$$g_j(t) = g_j(t; \theta_{1j}, \theta_{2j}) = \begin{cases} \dfrac{1}{\theta_{1j}} \exp\left\{-\dfrac{1}{\theta_{1j}} t\right\} & \text{if } 0 < t < \tau \\ \dfrac{1}{\theta_{2j}} \exp\left\{-\dfrac{1}{\theta_{1j}}\tau - \dfrac{1}{\theta_{2j}}(t-\tau)\right\} & \text{if } \tau \le t < \infty \end{cases} \tag{2.2}$$

for $j = 1, 2$. Since we will observe only the smaller of T_1 and T_2, let $T = \min\{T_1, T_2\}$ denote the overall failure time of a test unit. Then, its CDF and PDF are readily obtained to be

$$\begin{aligned} F(t) = F(t; \boldsymbol{\theta}) &= 1 - (1 - G_1(t))(1 - G_2(t)) \\ &= \begin{cases} 1 - \exp\left\{-\left(\dfrac{1}{\theta_{11}} + \dfrac{1}{\theta_{12}}\right) t\right\} & \text{if } 0 < t < \tau \\ 1 - \exp\left\{-\left(\dfrac{1}{\theta_{11}} + \dfrac{1}{\theta_{12}}\right)\tau - \left(\dfrac{1}{\theta_{21}} + \dfrac{1}{\theta_{22}}\right)(t-\tau)\right\} & \text{if } \tau \le t < \infty, \end{cases} \end{aligned} \tag{2.3}$$