# Signed line graphs with least eigenvalue -2 : The star complement technique 

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#### Abstract

We use star complement technique to construct a basis for -2 of signed line graphs using their root signed graphs. In other words, we offer a generalization of the corresponding results known in the literature for (unsigned) graphs in the context of line graphs and generalized line graphs.


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## 1. Introduction

Let $G=(V, E)$ be a simple graph of order $n=|V|$ and size $m=|E|$, and let $\sigma: E \rightarrow\{+,-\}$ be a mapping defined on the edge set of $G$. Then $\Gamma=(G, \sigma)$, or $G_{\sigma}$ for short, is a signed graph (or sigraph). The graph $G$ is its underlying graph, while $\sigma$ its sign function (or signature). An edge $e$ is positive (negative) if $\sigma(e)=+$ (resp. $\sigma(e)=-$ ). If all edges in $\Gamma=G_{\sigma}$ are positive (negative), then it is denoted by $G_{+}$(resp. $G_{-}$). Furthermore, it is common to interpret the signs as the integers $\pm 1$, so signed graphs can be also viewed as weighted graphs with these weights. Occasionally, we will allow multiple edges in underlying graphs, but not loops.

Actually, each concept defined for the underlying graph is transferred with signed graphs. For example, the degree of a vertex $v$ in $G$ is also its degree in $\Gamma$. Furthermore, if some subgraph of the underlying graph is observed, then the sign function for the subgraph is the restriction of the previous one.

The sign of cycle $C$ is defined as $\sigma(C)=\Pi_{e \in E(C)} \sigma(e)$, where $E(C)$ is the edge set of $C$. If $\sigma(C)>0(<0)$ then $C$ is positive (resp. negative). A signed graph is said to be balanced if all its cycles are positive; otherwise, it is unbalanced. For $\Gamma=G_{\sigma}$ and $U \subset V$, let $\Gamma^{U}$ be the signed graph obtained from $\Gamma$ by reversing the signature of the edges in the cut [ $U, V \backslash U$ ]. Namely, then $\sigma_{\Gamma^{U}}(e)=-\sigma_{\Gamma}(e)$ for any edge $e$ between $U$ and $V \backslash U$, and $\sigma_{\Gamma^{U}}(e)=\sigma_{\Gamma}(e)$ otherwise. The signed graph $\Gamma^{U}$ is said to be (signature) switching equivalent to $\Gamma$. In fact, switching equivalent signed graphs can be considered as (switching) isomorphic graphs and their signatures are said to be equivalent. It is easy to see that switching preserves the cycles signs.

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Fig. 1. Construction of a generalized line graph.


Fig. 2. Bi-directed edges.
Recall, the line graph of $G$, denoted by $\mathcal{L}(G)$, is the graph whose vertices are the edges of $G$, with two vertices of $\mathcal{L}(G)$ being adjacent whenever the corresponding edges in $G$ have just one vertex in common. Let $C P(k)$ be a cocktail-party graph on $2 k$ vertices, i.e., the graph obtained from the complete graph on $2 k$ vertices by removing from it a perfect matching.

The generalized line graph [10] is obtained in the following way from $G$ and an $n$-tuple $a=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ of non-negative integers assigned to its vertices: we take a copy of $\mathcal{L}(G)$ and a collection $C P\left(a_{1}\right), C P\left(a_{2}\right), \ldots, C P\left(a_{n}\right)$ of disjoint copies of cocktail-party graphs ( $C P\left(a_{i}\right)$ corresponds to vertex $v_{i}$ ), and then join by an edge each vertex of $C P\left(a_{i}\right)$ with vertices of $\mathscr{L}(G)$ that correspond to the edges of $G$ incident to $v_{i}$. An alternative way of constructing a generalized line graph from $(G ; a)$ is to consider the multigraph $G_{a}$ obtained from $G$ by adding to each vertex $v_{i}$ (of $\left.G\right) a_{i}$ petals, i.e., $a_{i}$ double pendant edges. Note here that two edges belonging to the same petal in $G_{a}$ correspond to non-adjacent vertices in $\mathcal{L}(G ; a)$. In Fig. 1 we depict a construction of generalized line graph. Now, the multigraph $G_{a}$ is regarded as a root graph of the generalized line graph; so $\mathscr{L}\left(G_{a}\right)=\mathscr{L}(G ; a)$. Clearly, $\mathscr{L}(G)=\mathscr{L}\left(G_{a}\right)$ if all entries of $a$ are zero. For further details, especially in the context of spectral graph theory, the reader is referred to [6]. For a (possibly) complete bibliography on signed graphs, we refer the reader to [13]. For a notation not given here, we refer the reader to [4,7] and [12].

Before providing the "signed" counterparts of results from [7], we need some further preparatory facts.
Let $A(G)=\left(a_{i j}\right)$ be the usual adjacency matrix of a (multi)graph $G$. Recall, $a_{i j}$ stands for the number edges joining vertices $i$ and $j$. If weighted (multi)graphs are considered, then $a_{i j}$ turns to be the sum of weights of all edges between vertices $i$ and $j$. In particular, if $\Gamma$ is just a signed graph, then $A(\Gamma)=\left(a_{i j}^{\sigma}\right)$, where $a_{i j}^{\sigma}=\sigma(i j) a_{i j}$.

We will also need the concept of bi-directed graphs. Let $\eta: V \times E \rightarrow\{-1,0,+1\}$ be a function which assigns to each pair $(v, e) \in V \times E$, provided $v$ is incident to $e$, an arrow directed to $v$ if $\eta(v, e)=+1$, or directed out of $v$ if $\eta(v, e)=-1$ (see Fig. 2); clearly, $\eta(v, e)=0$ if $v$ is not incident to $e$. In addition, the concept of oriented (resp. unoriented) edges are described in the same figure. It is easy to check that for any edge $e \in E$, where $e=u v$, the following condition holds

$$
\begin{equation*}
\sigma(e)=-\eta(u, e) \eta(v, e) \tag{1}
\end{equation*}
$$

Clearly, each bi-directed graph corresponds to a unique signed graph but not vice versa (as can be easily seen from (1)). Namely, in bi-directed graphs for each edge we can exchange the direction of arrows in it without affecting the sign in the corresponding sign graph. In some considerations, this multi-choice of arrow directions within edges can be important, and therefore we will occasionally denote $\Gamma$ by $\Gamma_{\eta}$.

The incidence matrix of $\Gamma$ with respect to $\eta$ is defined as follows

$$
B\left(\Gamma_{\eta}\right)=\left(b_{i j}\right)_{n \times m},
$$

where $b_{i j}=\eta\left(v_{i}, e_{j}\right)$.
We can next easily check that

$$
\begin{equation*}
B\left(\Gamma_{\eta}\right)^{\top} B\left(\Gamma_{\eta}\right)=2 I+A\left(\mathscr{L}\left(\Gamma_{\eta}\right)\right), \tag{2}
\end{equation*}
$$

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