Discrete Applied Mathematics 207 (2016) 80–89

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

(Total) Vector domination for graphs with bounded branchwidth[☆,☆☆]

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ARTICLE INFO

Article history:
Received 10 July 2015
Received in revised form 2 March 2016
Accepted 11 March 2016
Available online 4 April 2016

Keywords:
Vector dominating set
Total vector dominating set
Multiple dominating set
FPT
Bounded branchwidth

ABSTRACT

Given a graph $G = (V, E)$ of order n and an n-dimensional non-negative vector $d = (d(1), d(2), \ldots, d(n))$, called demand vector, the vector domination (resp., total vector domination) is the problem of finding a minimum $S \subseteq V$ such that every vertex v in $V \setminus S$ (resp., in V) has at least $d(v)$ neighbors in S. The (total) vector domination is a generalization of many dominating set type problems, e.g., the dominating set problem, the k-tuple dominating set problem (this k is different from the solution size), and so on, and its approximability and inapproximability have been studied under this general framework. In this paper, we show that a (total) vector domination of graphs with bounded branchwidth can be solved in polynomial time. This implies that the problem is polynomially solvable also for graphs with bounded treewidth. Consequently, the (total) vector domination problem for a planar graph is subexponential fixed-parameter tractable with respect to k, where k is the size of solution.

1. Introduction

Given a graph $G = (V, E)$ of order n and an n-dimensional non-negative vector $d = (d(1), d(2), \ldots, d(n))$, called *demand vector*, the *vector domination* (resp., *total vector domination*) is the problem of finding a minimum $S \subseteq V$ such that every vertex v in $V \setminus S$ (resp., in V) has at least $d(v)$ neighbors in S. These problems were introduced by [21], and they generalize many existing problems, such as the minimum dominating set and the k-tuple dominating set problem (this k is different from the solution size) [22,23], and so on. Indeed, by setting $d = (1, \ldots, 1)$, the vector domination becomes the minimum dominating set, and by setting $d = (k, \ldots, k)$, the total vector dominating set becomes the k-tuple dominating set. If in the definition of total vector domination, we replace open neighborhoods with closed ones, we get the *multiple domination*. In this paper, we sometimes refer to these problems just as *domination problems*. Table 1 of [8] summarizes how different variants of domination problems relate to one another. We also refer the interest reader to [23,24] for different applications of domination problems.

☆ An extended abstract of this article was presented in *Proceedings of the 11th Latin American Theoretical Informatics Symposium* (LATIN 2014), Lecture Notes in Computer Science, Vol. 8392, pp. 238–249, Springer, 2014.

☆☆ This work is partially supported by KAKENHI Nos. 23500022, 24700001, 24106004, 25104521, 25106508 and 26280001, the Kayamori Foundation of Informational Science Advancement and The Asahi Glass Foundation.

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http://dx.doi.org/10.1016/j.dam.2016.03.002
0166-218X/

Since the vector or multiple domination includes the setting of the ordinary dominating set problem, it is obviously NP-hard, and further it is NP-hard to approximate within $(c \log n)$-factor, where c is a positive constant, e.g., 0.2267 [1,26]. As for the approximability, since the domination problems are special cases of a set-cover type integer programming problem, it is known that the polynomial-time greedy algorithm achieves an $O(\log n)$-approximation factor [15] and it is already optimal in terms of order. For further analyses in terms of approximability and inapproximability see [9,8].

In this paper, we focus on another aspect of designing algorithms for domination problems, that is, the polynomial-time solvability of the domination problems for graphs of bounded treewidth or branchwidth. In [3], it is shown that the vector domination problem is $W[1]$-hard with respect to treewidth. On the other hand, from Courcelle's meta-theorem about MSOL [11], if the vector domination would be expressible in MSOL, then it would be linearly solvable for graphs with bounded treewidth, which contradicts the $W[1]$-hardness with respect to treewidth. To design a polynomial-time algorithm for the domination problems, we need to find a different way other than the typical MSOL approach.

In this paper, we present a polynomial-time algorithm for the domination problems in graphs with bounded branchwidth. The branchwidth is a measure of the "global connectivity" of a graph, and is known to be strictly related to treewidth. It is known that $\max\{bw(G), 2\} \le tw(G) + 1 \le \max\{3bw(G)/2, 2\}$, where $bw(G)$ and $tw(G)$ denote the branchwidth and treewidth of graph G, respectively [28]. Due to the linear relation of these two measures, polynomial-time solvability of a problem for graphs with bounded treewidth implies polynomial-time solvability of a problem for graphs with bounded branchwidth, and vice versa. Hence, our results imply that the domination problems (i.e., vector domination, total vector domination and multiple domination) can be solved in polynomial time for graphs with bounded treewidth; the polynomial-time solvability for all the problems (except the dominating set problem) in Table 1 of [8] is newly shown. Also, they answer the question by [9,8] about the complexity status of the domination problems of graphs with bounded treewidth.

Furthermore, by using the polynomial-time algorithms for graphs of bounded treewidth, we can show that these problems for a planar graph are subexponential fixed-parameter tractable with respect to the size of the solution k, that is, there is an algorithm whose running time is $2^{O(\sqrt{k}\log k)}n^{O(1)}$. To our best knowledge, these are the first fixed-parameter algorithms for the total vector domination and multiple domination, whereas the vector domination for planar graphs has been shown to be FPT [27]. For the latter case, our algorithm greatly improves the running time.

Note that the polynomial-time solvability of the vector domination problem for graphs of bounded treewidth has been independently shown very recently [7]. They considered a further generalization of the vector domination problem, and gave a polynomial-time algorithm for graphs of bounded clique-width. Since $cw(G) \le 3 \cdot 2^{tw(G)-1}$ holds where $cw(G)$ denotes the clique-width of graph G [10], their polynomial-time algorithm implies the polynomial-time solvability of the vector domination problem for graphs of bounded treewidth and bounded branchwidth.

1.1. Related work

For graphs with bounded treewidth (or branchwidth), the ordinary domination problems can be solved in polynomial time. As for the fixed-parameter tractability, it is known that even the ordinary dominating set problem is W[2]-complete with respect to solution size k, and hence it is unlikely to be fixed-parameter tractable [17]. In contrast, it can be solved in $O(2^{11.98\sqrt{k}}k + n^3)$ time for planar graphs, that is, it is subexponential fixed-parameter tractable [16]. The subexponential part comes from the inequality $bw(G) \le 12\sqrt{k}+9$, where k is the size of a dominating set of G. Behind the inequality, there is a unified property of parameters, called *bidimensionality* [14]. Namely, the subexponential fixed-parameter algorithm of the dominating set for planar graphs (more precisely, H-minor-free graphs [13]) is based on the bidimensionality.

A maximization version of the ordinary dominating set is also considered. *Partial Dominating Set* is the problem of maximizing the number of vertices to be dominated by using a given number k of vertices. In [2], it was shown that partial dominating set problem is FPT with respect to k for H-minor-free graphs. Later, [18] gives a subexponential FPT with respect to k for apex-minor-free graphs, also a superclass of planar graphs. Although partial dominating set is an example of problems to which the bidimensionality theory cannot be applied, the authors of [18] develop a technique to reduce an input graph so that its treewidth becomes $O(\sqrt{k})$.

For the vector domination, a polynomial-time algorithm for graphs of bounded treewidth has been proposed very recently [7], as mentioned before. In [27], it is shown that the vector domination for ρ-degenerated graphs can be solved in $k^{O(\rho k^2)}n^{O(1)}$ time, if $d(v) > 0$ holds for $\forall v \in V$ (positive constraint). Since any planar graph is 5-degenerated, the vector domination for planar graphs is fixed-parameter tractable with respect to solution size, under the positive constraint. Furthermore, the case where $d(v)$ could be 0 for some v can be easily reduced to the positive case by using the transformation discussed in [3], while increasing the degeneracy by at most 1. It follows that the vector domination for planar graphs is FPT with respect to solution size k. However, for the total vector domination and multiple domination, neither polynomial time algorithm for graphs of bounded treewidth nor fixed-parameter algorithm for planar graphs has been known.

Other than these, several generalized versions of the dominating set problem are also studied. (k, r)-center problem is the one that asks the existence of set S of k vertices satisfying that for every vertex $v \in V$ there exists a vertex $u \in S$ such that the distance between u and v is at most r; $(k, 1)$-center corresponds to the ordinary dominating set. The (k, r)-center for planar graphs is shown to be fixed-parameter tractable with respect to k and r [12]. For $\sigma, \rho \subseteq \{0, 1, 2, \ldots\}$ and a positive integer k, $\exists[\sigma, \rho]$-dominating set is the problem that asks the existence of set S of k vertices satisfying that $|N(v) \cap S| \in \sigma$