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Matching preclusion for vertex-transitive networks[☆]

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ABSTRACT

In interconnection networks, matching preclusion is a measure of robustness in the event of link failure. Let G be a graph of even order. The matching preclusion number $mp(G)$ is defined as the minimum number of edges whose deletion results in a graph without perfect matchings. Many interconnection networks are super matched, that is, their optimal matching preclusion sets are precisely those induced by a single vertex. In this paper, we obtain general results of vertex-transitive graphs including many known networks. A k-regular connected vertex-transitive graph of even order has matching preclusion number k and is super matched except for six classes of graphs. From this many results already known can be directly obtained and matching preclusion for some other networks, such as folded k-cube graphs, Hamming graphs and halved k-cube graphs, are derived.

1. Introduction

A *network* (or *graph*) is a collection of points or nodes, called *vertices*, and a collection of links, called *edges*, each connecting two nodes. The number of vertices of a graph G is its *order*, written as $|G|$; its number of edges is denoted by $\|G\|$. We refer $V(G)$ and $E(G)$ as the vertex set and edge set of G respectively. The matching preclusion, viewed as a measure of the robustness of graphs, of many networks have been investigated. By summarizing these results, we can see that almost all the networks considered are vertex-transitive and surprisingly, their matching preclusion almost act in the same way. A natural question arises: What does the matching preclusion of vertex-transitive graphs act? More precisely, can we obtain a unified property on the matching preclusion of vertex-transitive graphs?

Let G be a graph of even order. A *perfect matching* of G is a set of edges such that every vertex is incident with exactly one edge in this set. For $S \subseteq E(G)$, if $G-S$ has no perfect matchings, where $G-S$ denotes the subgraph of G by deleting S from G, then we call S a *matching preclusion set*. The *matching preclusion number* of G, denoted by $mp(G)$, is the minimum cardinality among all matching preclusion sets. Correspondingly, the matching preclusion set attaining the matching preclusion number is called an *optimal matching preclusion set* (or in short, *optimal solution*). The concept of matching preclusion was introduced by Brigham et al. for "measuring the robustness of a communications network graph which is a model for the distributed algorithm that requires each node of it to be matched with a neighboring partner node" [1].

Until now, the matching preclusion numbers of lots of networks (graphs) have been computed, such as the Petersen graph, hypercubes, complete graphs and complete bipartite graphs [1], Cayley graphs generalized by transpositions and (n, k)-star graphs [5], augmented cubes [8], (n, k)-bubble-sort graphs [7], tori and related Cartesian products [6], burnt

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pancake graphs [9], balanced hypercubes [11], restricted HL-graphs and recursive circulant $G(2^m, 4)$ [14], and k-ary n-cubes [18]. Their optimal solutions have also been classified.

By deleting the edges incident with a given vertex in a graph, the resulting subgraph has no perfect matchings. Hence the matching preclusion number is bounded by the minimum degree.

Theorem 1.1 ([5]). *Let G be a graph of even order. Then $mp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

In a network, a vertex with a special matching vertex at any time implies that tasks running on a fault vertex can be shifted onto its matching vertex. Thus under this fault assumption, larger $mp(G)$ signifies higher fault tolerance. Fortunately, matching preclusion numbers of many regular interconnection networks of degree k attain the maximum value, the minimum degree k [4]. Moreover, the optimal solutions are precisely those induced by a single vertex except for the ones of small order. Formally, we call the optimal solution induced by a single vertex a *trivial optimal solution* (*non-trivial optimal solution* otherwise) and the graphs with all optimal solutions trivial *super matched*. Generally, in the event of a random link failure, it is very unlikely that all of the links incident to a single vertex fail simultaneously. From this point of view, that a graph is super matched implies that it has higher fault tolerance.

A graph H is called *vertex-transitive* if for any two vertices x, y in $V(H)$, there exists an automorphism φ of H such that $\varphi(x) = y$. Recalling that many of the networks whose matching preclusion have been considered are vertex-transitive graphs and almost all of them are super matched. Fortunately, we obtain that almost all vertex-transitive graphs have such properties, too. Precisely, we get the following result, in which, $Z_{4n}(1, 4n-1, 2n)$ stands for the Cayley graph on Z_{4n}, the additive group modulo $4n$, with the generating set $S = \{1, 4n-1, 2n\}$. $Z_{4n+2}(2, 4n, 2n+1)$ and $Z_{4n+2}(1, 4n+1, 2n, 2n+2)$ are defined similarly.

Theorem 1.2. *A k-regular connected vertex-transitive graph G of even order is super matched if and only if it does not contain cliques of size k when k is odd and $k \leq |G| - 2$ or it is not isomorphic to a cycle of length at least six or $Z_{4n}(1, 4n-1, 2n)$ or $Z_{4n+2}(2, 4n, 2n+1)$ or $Z_{4n+2}(1, 4n+1, 2n, 2n+2)$ or the Petersen graph.*

This article is organized as follows. In Section 2, we will analyze some structural properties of vertex-transitive graphs. In Section 3, we present the proof of Theorem 1.2. In Section 4, we make a conclusion and several applications to the matching preclusion of some networks.

2. Preliminaries

In this section, we shall present several results that will be used later. An edge set $S \subseteq E(G)$ is called an *edge-cut* if there exists a set $X \subseteq V(G)$ such that S is the set of edges between X and $\overline{X}$, where $\overline{X} := V(G) \setminus X$. The *edge-connectivity* $\lambda(G)$ of G is the minimum cardinality over all edge-cuts of G. Mader proved the following result.

Theorem 2.1 ([12]). *If G is a k-regular connected vertex-transitive graph, then $\lambda(G) = k$.*

The following lemma makes a step further by characterizing the minimum edge-cuts of vertex-transitive graphs, where a clique of a graph G is a subset of $V(G)$ such that every two vertices in it are adjacent in G.

Theorem 2.2 ([10], Lemma 5.5.26). *Let G be a k-regular connected vertex-transitive graph. Then $\lambda(G) = k$ and either*

(i) *every minimum edge-cut of G is the star of a vertex, or*

(ii) *G arises from a (not necessarily simple) vertex- and edge-transitive k-regular graph G_0 by a k-clique (a clique of size k) insertion at each vertex of G_0. Moreover, every minimum edge-cut of G is the star of a vertex of G or a minimum edge-cut of G_0.*

The following corollary that will be used in Section 3 follows immediately. An edge-cut is called *trivial* if it isolates a vertex and *non-trivial* otherwise.

Corollary 2.3. *For a k-regular connected vertex-transitive graph G, every k-edge-cut (an edge-cut of size k) is either trivial or the deletion of it results in two components, and the vertex set of each component is partitioned into several k-cliques.*

For a k-regular graph G, if every minimum edge-cut of it is trivial, then we say it is *super-edge-connected* (or simply *super-λ*). For vertex-transitive graphs, J. Meng has presented a characterization with respect to the cliques.

Theorem 2.4 ([13]). *Let G be a k-regular connected vertex-transitive graph which is neither a complete graph nor a cycle. Then G is super-λ if and only if it does not contain k-cliques.*

Theorem 2.4 is used to characterize the structure of 3-regular connected non-bipartite vertex-transitive graphs with respect to the length of minimum odd cycles. As we will see, minimum odd cycles play a crucial role in the following proofs. Here we make a convention that is suitable throughout this paper. For a cycle drawn on the plane without crossings, let $a, b \in V(C)$, denote $C(a, b)$ by the subgraph of C from a to b along a clockwise direction. A cycle C is called a *minimum odd cycle* if $\|C\|$ is odd and the minimum among lengths of all odd cycles. For a minimum odd cycle, we usually say it is *minimum*. The following two results (Lemmas 2.5 and 2.6) will be used to prove Lemma 2.7, which plays an important role in the proof of Theorem 1.2 in the next section. Herein and hereafter, we denote K_m by the complete graph with m vertices.