Discrete Applied Mathematics 207 (2016) 99–105

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On the minimum Kirchhoff index of graphs with a fixed number of cut vertices

Ashkan Nikseresht

Department of Mathematics, Institute for Advanced Studies in Basic Sciences, P.O. Box 45195-1159, Zanjan, Iran

ARTICLE INFO

Article history:
Received 7 June 2015
Received in revised form 15 February 2016
Accepted 23 February 2016
Available online 21 March 2016

Keywords:
Kirchhoff index
Resistance distance
Block decomposition

ABSTRACT

Consider the family $\mathcal{C}$ of all connected simple graphs on n vertices which have k cut-vertices. We state some properties of the graph in $\mathcal{C}$ which has the minimum Kirchhoff index. In particular, we characterize this graph in the cases that $k \le \frac{n}{2}$ or $k \ge n-3$. Also we characterize the graph G in $\mathcal{C}$ having a non-cut-vertex with the lowest resistive eccentricity index among all non-cut-vertices of all graphs in $\mathcal{C}$.

1. Introduction

In this paper, all graphs are supposed to be simple and connected. Also, we assume that G is a graph with vertex set $\mathrm{V}(G)$ of size $n(G)$ and edge set $\mathrm{E}(G)$ and by $\mathrm{N}(x)$ we mean the set of neighbors of a vertex x.

The concept of resistance distance was first introduced by Klein and Randić [8]. If we view G as an electrical network and replace each edge of G with a unit resistor, the *resistance distance* between any two vertices a and b, denoted by $r_{ab}(G)$ or simply r_{ab}, is defined to be the effective resistance between a and b as computed by Ohm's and Kirchhoff's laws (for a detailed definition see [12]). Now the *Kirchhoff index* of G is defined as:

$$\mathrm{Kf}(G) = \frac{1}{2} \sum_{a,b \in \mathrm{V}(G)} r_{ab}.$$

Also $\mathrm{Kf}_v(G) = \sum_{a \in \mathrm{V}(G)} r_{va}$ is called the *resistive eccentricity index of* v.

Recently this concept has got a wide attention from different authors especially those interested in applications in quantum chemistry, see for example [2,4,11]. In some researches such as [1,15,14], resistance distance is studied from a probabilistic point of view and a relation between this concept and Markov chains is established. Many researchers have paid attention to the Kirchhoff index of graphs with some symmetry or graphs constructed from graph operations, for example see [7,10,5,6,17,19,22,21,13]. Another question which has got attention recently is finding graphs having minimum or maximum Kirchhoff index within a specific class of graphs, see [9,3,18,20,16]. In particular, in [3] the minimum Kirchhoff index of graphs with a given number of cut-edges is computed.

The aim of this work is to study the graphs which have the minimum Kirchhoff index in the class of all graphs on n vertices with k cut-vertices for fixed integers $0 \le k \le n-2$ (we denote this class by $\mathcal{C}_{n,k}$ or $\mathcal{C}$). To this end, first in Section 2, we find the graph in $\mathcal{C}$ with a vertex having minimum resistive eccentricity among all non-cut-vertices of all graphs in $\mathcal{C}$. Then in Section 3, we state some properties of the graphs with the minimum Kirchhoff index in $\mathcal{C}$. As a consequence of these properties, we characterize this graph uniquely when $k \le \frac{n}{2}$.

E-mail address: ashkan_nikseresht@yahoo.com.

http://dx.doi.org/10.1016/j.dam.2016.02.022
0166-218X/

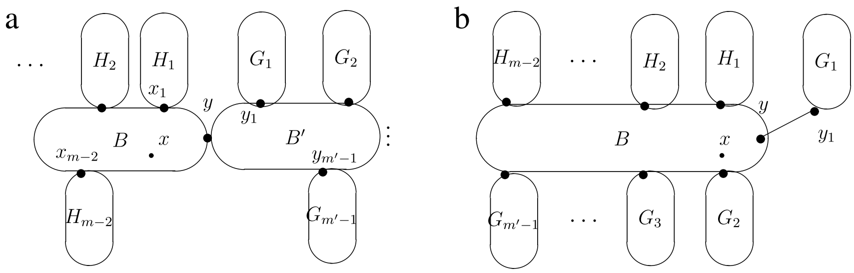

Fig. 1. Illustrations for proof of 2.4: (a) G; (b) G'.

2. Minimum resistive eccentricity index of non-cut-vertices in $\mathcal{C}$

We start with some lemmas. Assume that x is a cut-vertex and H_1 and H_2 are subgraphs of G. Here we say that (H_1, H_2) is an *x-separation* of G, when $G = H_1 \cup H_2$ and $\{x\} = \mathrm{V}(H_1) \cap \mathrm{V}(H_2)$.

Lemma 2.1 ([8, Lemma E]). *Let x be a cut-vertex of a graph G and (H_1, H_2) be an x-separation of G. If $i \in \mathrm{V}(H_1)$ and $j \in \mathrm{V}(H_2)$, then $r_{ij}(G) = r_{ix}(H_1) + r_{xj}(H_2)$.*

Lemma 2.2 ([12, Lemma 2.2]). *Let x be a cut-vertex of G and (H_1, H_2) be an x-separation of G. If $i, j \in \mathrm{V}(H_1)$, then $r_{ij}(G) = r_{ij}(H_1)$.*

Lemma 2.3. *If K_n is the complete graph on n vertices, then $\mathrm{Kf}(K_n) = n - 1$, $\mathrm{Kf}_x(K_n) = 2 - \frac{2}{n}$ and $r_{xy} = \frac{2}{n}$ for each $x, y \in \mathrm{V}(K_n)$.*

Proof. Easy and well-known. ■

Assume $G \in \mathcal{C}_{n,k}$. We say that a non-cut-vertex $x \in \mathrm{V}(G)$ *attains minimum resistive eccentricity* (*MRE*, for short) on non-cut-vertices, when for each $H \in \mathcal{C}_{n,k}$ and each non-cut-vertex y of H, we have $\mathrm{Kf}_x(G) \le \mathrm{Kf}_y(H)$. Also if B is a block of G, we sometimes misuse the notation and call $G[B]$ a block of G, too.

Proposition 2.4. *Suppose that $G \in \mathcal{C}$ and $x \in \mathrm{V}(G)$ attains MRE on non-cut-vertices. Then every block of G not containing x is an edge.*

Proof. Let B be the block which contains x (note that x is a non-cut-vertex and hence is contained in just one block) and set $m = n(B)$. Suppose that G has a block $B' \neq B$ which is not an edge. Then $m' = n(B') > 2$. We will construct a graph $G' \in \mathcal{C}_{n,k}$ by replacing some vertices of G, for which $\mathrm{Kf}_x(G') < \mathrm{Kf}_x(G)$, which is contrary to the assumption. Since adding edges decreases $\mathrm{Kf}_x(G)$, each block of G is a clique. Also we can assume that each cut-vertex c is contained in exactly two blocks, else by adding all edges between two of the blocks containing c, we get a graph in $\mathcal{C}_{n,k}$ while decreasing $\mathrm{Kf}_x(G)$.

First suppose that B and B' have a common cut-vertex y. By the above assumptions the only blocks containing y are B and B'. Let y_i for $1 \le i \le m' - 1$ [x_j for $1 \le j \le m - 2$] be the vertices of B' [B] different from y [from x and y]. Set G_i [H_j] to be the component of the graph obtained by deleting B' [B] – as an edge set – which contains y_i [x_j], and $n_i = n(G_i)$ [$r_j = n(H_j)$] (see Fig. 1(a)). We may assume $n_1 \le n_2 \le \cdots \le n_{m'-1}$. Note that the sum of all r_{xa}'s with $a \in \mathrm{V}(B') \setminus \{y\}$ is $(m' - 1)r_{xy} + \mathrm{Kf}_y(B')$ by 2.1. Using 2.2 and 2.3 and a similar argument for the cases $a \in \mathrm{V}(G_i) \setminus \{y_i\}$ or $a \in \mathrm{V}(H_j) \setminus \{x_j\}$ we see that

$$\mathrm{Kf}_x(G) = 2 - \frac{2}{m} + 2 - \frac{2}{m'} + (m' - 1)\frac{2}{m} + \sum_{i=1}^{m'-1}\left(\mathrm{Kf}_{y_i}(G_i) + (n_i - 1)\left(\frac{2}{m} + \frac{2}{m'}\right)\right) + \sum_{j=1}^{m-2}\left(\mathrm{Kf}_{x_j}(H_j) + (r_j - 1)\frac{2}{m}\right).$$

Now let G' be the graph obtained from G by 'detaching' all G_i for $2 \le i \le m' - 1$ from B' and 'attaching' them to B, that is, by deleting all edges $y_1 y_i$ for $2 \le i \le m' - 1$ and adding all edges between each y_i and vertices of B for $2 \le i \le m' - 1$ (see Fig. 1(b)). Note that in G' the block containing x is a clique on $m + m' - 2$ vertices and the other block containing y is an edge. Similar to $\mathrm{Kf}_x(G)$ we can calculate $\mathrm{Kf}_x(G')$:

$$\mathrm{Kf}_x(G') = 2 - \frac{2}{m + m' - 2} + 1 + \frac{2}{m + m' - 2} + \sum_{i=1}^{m'-1}\left(\mathrm{Kf}_{y_i}(G_i) + (n_i - 1)\frac{2}{m + m' - 2}\right) + (n_1 - 1) + \sum_{j=1}^{m-2}\left(\mathrm{Kf}_{x_j}(H_j) + (r_j - 1)\frac{2}{m + m' - 2}\right).$$