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Eccentricity sums in trees

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ABSTRACT

The eccentricity of a vertex, $\text{ecc}_T(v) = \max_{u\in T} d_T(v,u)$, was one of the first, distance-based, tree invariants studied. The total eccentricity of a tree, $\text{Ecc}(T)$, is the sum of the eccentricities of its vertices. We determine extremal values and characterize extremal tree structures for the ratios $\text{Ecc}(T)/\text{ecc}_T(u)$, $\text{Ecc}(T)/\text{ecc}_T(v)$, $\text{ecc}_T(u)/\text{ecc}_T(v)$, and $\text{ecc}_T(u)/\text{ecc}_T(w)$ where u, w are leaves of T and v is in the center of T. In addition, we determine the tree structures that minimize and maximize total eccentricity among trees with a given degree sequence.

1. Introduction

The *eccentricity* of a vertex v in a connected graph G is defined in terms of the distance function as

$$\text{ecc}_G(v) := \max_{u\in V(G)} d(u,v).$$

The radius of G, $\text{rad}(G)$, is the minimum eccentricity while the diameter, $\text{diam}(G)$, is the maximum. The center, $C(G)$, is the collection of vertices whose eccentricity is exactly $\text{rad}(G)$.

We focus our attention on trees, where the center has at most two vertices [7] and the diameter is realized by a leaf. We also explore the *total eccentricity* of a tree T, defined as the sum of the vertex eccentricities:

$$\text{Ecc}(T) := \sum_{z\in V(T)} \text{ecc}_T(z).$$

For a fixed tree T with $v \in C(T)$ and any $z \in V(T)$,

$$\min_{u\in L(T)} \frac{\text{Ecc}(T)}{\text{ecc}_T(u)} \le \frac{\text{Ecc}(T)}{\text{ecc}_T(z)} \le \frac{\text{Ecc}(T)}{\text{ecc}_T(v)}$$

where $L(T)$ denotes the leaf set of T. This motivates the study in Section 2 of the extremal values and structures for the following ratios where $u, w \in L(T)$ and $v \in C(T)$,

$$\frac{\text{Ecc}(T)}{\text{ecc}_T(v)}, \quad \frac{\text{Ecc}(T)}{\text{ecc}_T(u)}, \quad \frac{\text{ecc}_T(u)}{\text{ecc}_T(v)}, \quad \text{and} \quad \frac{\text{ecc}_T(u)}{\text{ecc}_T(w)}.$$

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The results are analogous to similar studies on distance in [2] and on the number of subtrees in [11,10]. As in those papers, the behavior of ratios is more delicate than that of their numerators or denominators.

For a graph with n vertices, the total eccentricity is n times the average eccentricity. Dankelmann and Mukwembi [6] gave sharp upper bounds on the average eccentricity of graphs in terms of independence number, chromatic number, domination number, as well as connected domination number. For trees with n vertices, Dankelmann, Goddard, and Swart [5] showed that the path maximizes $\mathrm{Ecc}(T)$. In Section 3, we prove that the star minimizes $\mathrm{Ecc}(T)$ among trees with a given order. Turning our attention to trees with a fixed degree sequence, we prove that the "greedy" caterpillar maximizes $\mathrm{Ecc}(T)$ while the "greedy" tree minimizes $\mathrm{Ecc}(T)$. This provides further information about the total eccentricity of "greedy" trees across degree sequences.

From here forward, we assume that T is a tree with n vertices. Given two vertices $a, b \in V(T)$, $P(a, b)$ will be the unique path between a and b in T.

2. Extremal ratios

In this section, we consistently use the letters u, w to denote leaf vertices while v is a center vertex. Before delving into ratios, the following observation from [7] is given without proof, and will be used many times. The next observation is a simple calculation which will be useful in our proofs.

Observation 1. *The center, $C(T)$, contains at most 2 vertices. These vertices are located in the middle of a maximum length path, P. If $\{v\} = C(T)$, v divides P into two paths, each of length $\mathrm{rad}(T)$. If $\{v, z\} = C(T)$, the removal of $vz \in E(T)$ will divide P into two paths, each of length $\mathrm{rad}(T) - 1$.*

Observation 2. *For any path P with y edges and $y + 1$ vertices,*

$$\mathrm{Ecc}(P) = \sum_{z \in V(P)} \mathrm{ecc}_P(z) = \begin{cases} \frac{3}{4}y^2 + y & \text{if } y \text{ is even} \\ \frac{3}{4}y^2 + y + \frac{1}{4} & \text{if } y \text{ is odd.} \end{cases}$$

2.1. On the extremal values of $\frac{\mathrm{Ecc}(T)}{\mathrm{ecc}_T(v)}$ where $v \in C(T)$

For any tree T with v in the center, we determine the maximum and minimum values that the ratio $\frac{\mathrm{Ecc}(T)}{\mathrm{ecc}_T(v)}$ can achieve and characterize the trees for which these bounds are tight.

Theorem 3. *Let T be a tree with $n \geq 2$ vertices. For any $v \in C(T)$, we have*

$$\frac{\mathrm{Ecc}(T)}{\mathrm{ecc}_T(v)} \leq 2n - 1.$$

For $n \geq 3$, equality holds if and only if T is a star centered at v.

Proof. Let T be an arbitrary tree with $v \in C(T)$. It is known that for any tree T, $\mathrm{diam}(T) \leq 2\,\mathrm{rad}(T)$ and for any vertex $z \in V(T)$, $\mathrm{rad}(T) \leq \mathrm{ecc}_T(z) \leq \mathrm{diam}(T)$. Because $\mathrm{ecc}_T(v) = \mathrm{rad}(T)$, the bound in the theorem is proved as follows:

$$\mathrm{Ecc}(T) \leq \mathrm{ecc}_T(v) + (n-1)\,\mathrm{diam}(T) \leq (2n-1)\,\mathrm{rad}(T).$$

Equality holds precisely when T has $\mathrm{ecc}_T(z) = 2\,\mathrm{ecc}_T(v)$ for all vertices $z \neq v$. Because the eccentricities of adjacent vertices differ by at most 1, $\mathrm{ecc}_T(v) = 1$ and $\mathrm{ecc}_T(z) = 2$ for all $z \neq x$ which is only true for the star. □

Theorem 4. *Let T be a tree with $n \geq 2$ vertices. Let k and i be nonnegative integers with $0 \leq i \leq 2k$ and $n = k^2 + i$. For any $v \in C(T)$, we have*

$$\frac{\mathrm{Ecc}(T)}{\mathrm{ecc}_T(v)} \geq \begin{cases} n - 3 + 2k + \frac{i}{k} & \text{if } 0 \leq i \leq k \\ n - 3 + 2k + \frac{i+1}{k+1} & \text{if } k+1 \leq i \leq 2k. \end{cases}$$

For $n \geq 4$, equality holds if and only if T is a tree whose longest path has $2x$ vertices ($x = k$ in the first case and $x = k + 1$ in the second) and each other vertex is adjacent to one of the two center vertices of this path. For $i = k$, the two bounds agree and both values for x provide an extremal tree.

Proof. Let T be a tree with $n \geq 3$ vertices and let $v \in C(T)$. If T is a star then $\frac{\mathrm{Ecc}(T)}{\mathrm{ecc}_T(v)} = 2n - 1$ which is strictly greater than the bounds in the theorem.